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THE GRAVITY FIELD IN FENNOSCANCIA AND POSTGLACIAL CRUSTAL MOVEMENTS

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by

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CONTENT.

1. ON THE STRATEGY OF THE INVESTIGATION.

The present study is devoted to the problems associated with disposal of radioactive waste in the Swedish crust. Recent investigations seem to indicate that the present uplift in Fennoscandia is no longer of isostatic origin. A 'tectonic factor' has been detected and this component is supposed to dominate the uplift now recorded. If these findings are conclusive, then there is a potential risk that heavy stresses are building up in the Scandinavian crust and it should be extremely risky to use the crust for waste disposal. We recollect that the maximal uplift inside Sweden amounts to approximately 1 cm per year. This is a movement of the same magnitude as the corresponding vertical movements in the Thingvallier area on Iceland and the s:t Andrea´s fault in USA. We know that the tectonic movements in these two areas are linked with severe stresses (in the crust) that originate serious earthquakes. This means that if we have a similar strong tectonic movement in the Scandinavian crust, then no waste disposal should be considered in this crust. However, if the movements are caused by the isostatic compensation after the last glaciation in Fennoscandia, then the situation is much more favourable and waste disposal can be considered if suitable disposal areas can be found. We summarize:

1. If present movements are mainly tectonic, then we cannot recommend waste disposal in the Swedish crust. (The situation cannot be considered stationary.)
2. If present movements are mainly isostatic, then we can consider waste disposal in the Swedish crust as an acceptable alternative.

Isostatic and tectonic vertical movements have been announced from shoreline observations in NW England, West coast of Sweden, Blekinge and east coast of Sweden, and all these areas were linked to the Fennoscandian glaciation. All these findings are closely correlated with the definition of eustatic change of ocean sea level and we will use the following strategy for our analysis.

1. The technique for the determination of eustatic change will be be analysed.
2. The isostatic-tectonic factors in the following areas will be analysed.
   2.1 NW England
   2.2 Western Sweden
   2.3 Eastern Sweden
3. The geopotential field from the earth will be used for a determination of the residual gravity field in Fennoscandia, that has an unrelaxed isostatic component. This study will include two sections.
   3.1 Global data. (Satellite observations.)
   3.2 Local terrestrial data

All the findings from these six partial investigations will be used for a study of the present vertical movements of the Scandinavian crust. The result of these studies will be used for our conclusions concerning waste disposal in this crust.

2. THE ORIGIN OF UPLIFTS.

Movements in the Scandinavian crust are closely related with the glaciation history of this area. When the ice begins melting, then the meltwater is added to the oceans and the total water volume is changed. The classical method of analysing this uplift of mean sea level has been to compute "the uniform change" of the ocean depth for a "rigid earth". This procedure anticipates that there is a constant change all over the oceans (and the water connected with the oceans). The beaches in the glaciated zones are nowadays above the present sea level and observations of the shore line displacements have given the basis of former analysis of the uplift of the crust. If we are satisfied with a crude interpretation, then the apparent uplift of the shore is a direct measure of the uplift of the crust.

The apparent uplift of the crust is obtained when the mean sea level is determined at a selected time span and the differences in height above sea level are recorded with the use of repeated observations for a selected point.

The apparent uplift in the Scandinavian area has been studied by Lidén, Mörner, Sauramo and others. It is of course not correct to claim that the apparent uplift of the crust is identical with the true uplift of the crust. The true uplift of the crust, is the vertical displacement of the crust along the plumb line (in a crude approach the change in radius of the earth at the selected point).

When the ice melts away from the glaciated area, then principal consequences can be recorded:

1. Mean sea level of the oceans will rise (eustatic change).
2. The local sea level will subside in the glaciated areas and their surroundings. This is an instantaneous change that follows in the same moment as the melting (local geoidal subsidence).

3. The crust will rise in the formerly glaciated areas and in the surroundings. There is a primary fast elastic uplift which follows almost immediately combined with a slow viscoelastic uplift.
4. A slow viscous (or viscoelastic) movement then follows for thousands of years. This movement has often been called isostatic.
5. The increased load of the flooded areas will cause a secondary elastic subsidence of the crust. This subsidence follows rapidly as an elastic compensation.
6. The movements of mean sea level are superimposed on the crustal movements. They have a natural extension inside the continents. The geoid is the equipotential surface of the earth that coincides with mean sea level of the oceans.
7. The sun and the moon originate tidal movements in the crust and the oceans.
8. Movements independent of the load mechanism will mostly have a thermal origin. (Tectonic movements.)

The natural tools for studying the uplift mechanism are of course found in the advanced potential theory for viscoelastic bodies. A rigorous analysis will also have to consider that the crust should have purely elastic properties in an area, where no faults can be found. Modern studies of the crust indicate that the faults are so frequent that the elastic properties are only valid locally inside selected blocks. The Scancinavian gneiss plates are known to have a low fault rate. The remaining part of the Scandinavian crust is penetrated by faults from infinitesimal size up to the macro size. The large faults are normally restricted to well defined sites.

The elastic phase is dominating the ice melting period.

During this time, we can expect that the rapid uplift will have caused a multitide of severe faults that generated numerous earthquakes. However, the numerous glaciation periods in Scandinavia must have created a fault pattern that is so advanced, that a new glaciation period will have a limited impact on the total fault pattern.

3. THE EUSTATIC FACTOR IN THE UPLIFT.

Early studies of the Scandinavian uplift were restricted to a direct measurement of the apparent uplift. This means that the records refer to the difference in height (altitude) above sea level for a selected point on land. No analysis was made of the origin of this uplift. Studies of this type have been made by G. de Geer, R. Lidén and Sauramo. Mörner (1976) made a separate determination of the changes of ocean mean sea level (eustatic change) in order to get a better estimate of the true uplift of the crust. Using eight selected sites on the west coast of Sweden he made a determination of all detectable trangressions and regressions. (C^{14} dating.) The very interesting records of Mörner have not yet been analysed in a mathematical way and we will here make a tentative study of the observations. We note that several geologists have tried to make a determination of the "global" changes of mean sea level for selected time spans in the past. These global changes are often called "eustatic". We know that mean sea level of the oceans represents an equipotential surface of the earth. The total mass of the earth must remain constant when the ice over the arctic areas melts away. - The potential V_j of the earth can be written (with centrifugal potential)

$$V_j = (GM/r_j) \sum_{n=2}^{\infty} \sum_{m=0}^{n} (r_o/r_j)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} \sin\phi + d^2\omega^2/2$$

where ω is the rotational velocity, d the distance of the actual point to the axis of rotation, G gravitational constant, M mass of the earth, λ latitude, ϕ longitude r_o radius of the earth, r_j geocentric distance, c_{nm}, S_{nm} sperical harmonic constants and $P_{nm}(\sin\phi)$ Legendre polynomial. An equipotential surface is obtained by computing r_j for any ϕ and λ for a constant V_j. The meaning of "eustatic change" is not quite obvious in this context. If some arctic ice has melted away, then the geoid (equipotential surface coinciding with mean sea level) will rise in most parts of the oceans and mean r_j will increase by a small quantity. This can be an interpretation of the "mean eustatic" change. Then there is a "local eustatic change" of sea level valid for an area which is not subjected to crustal movements. There is of course no completely safe method of selecting an area without crustal movements. However, one can expect that outside an uplift area, there will be some rather "stable sites". If this is correct, then one should be able to measure the eustatic changes directly in these sites. If shore line records are available from a number of sites, then the "optimal estimate" of the eustatic change can be defined for this set of observations. We postulate that the crustal movements can be represented by a smooth mathematical function for the selected time span. The shore line records are considered to be outcomes from a number of stochastic processes $x(t)$, $y(t)$ ... and $z(t)$. We have the following outcomes from these stochastic processes:

Outcomes

$x(t_1), x(t_2), x(t_3) \ldots x(t_n)$

$y(t_1), y(t_2), y(t_3) \ldots y(t_n)$

....

$z(t_1), z(t_2), z(t_3) \ldots z(t_n)$

The stochastic processes include three fundamental sections

stochastic process = signal + noise + trend function.

There are n outcomes for each of N stochastic processes. We postulate that the same time spacing has been used for all stochastic processes in our study, but a generalization to arbitray time intervals is obvious.

Definition: The n-th order optimal estimator of the eustatic process from a number of given stocastic processes is the stochastic process which gives minimum pooled variance in estimating the regression curve of n-th order from the remaining stochastic processes subtracted by the optimal estimator.

We note that our definition is somewhat critical, because the selection of sites for the record has to be done with some care if a meaningful result should be expected. Furthermore, the choice of the regression curve is important. Exponential functions can of course also be contemplated or any more sophisticated trend function.

Hypothesis:

The stochastic processes $x(t)$, $y(t)$ ... $z(t)$ have the representation

$$x(t_i) = s(t_i) + e_i + c_1 t_i + c_2 t_i^2 \dots + c_n t_i^n$$

$$y(t_i) = s(t_i) + e_i' + c_1' t_i + c_2' t_i^2 \dots + c_n' t_i^n$$

....

$$z(t_i) = s(t_i) + e_i'' + c_1'' t_i + c_2'' t_i^2 \dots + c_n'' t_i^n$$

where

$s(t_i)$ = signal, common for all sites (stochastic process with unknown covariance function and not further specified) and valid for the time t_i

e_i, e_i', ... e_i'' = noise at the time t_i

c_1, c_2, etc = unknown parameters in a regression curve (trend function)

We have an unknown stochastic process which is common for all observations. This means that we can eliminate this common stochastic process by forming new auxiliary variables of the type

$$x(t_i) - y(t_i)$$

$$z(t_i) - y(t_i)$$

....

$$x(t_i) - z(t_i)$$

etc.

Furthermore, we introduce the new noise parameters

$$e_i' - e_i = v_i$$

$$e_i' - e_i'' = v_i'$$

etc

The c-parameters will be given a new meaning.

Solution:

We have for a second order approach.

Site: x

$$x(t_1) - y(t_1) + v_1 = c_1 t_1 + c_2 t_1^2$$

$$x(t_2) - y(t_2) + v_2 = c_1 t_2 + c_2 t_2^2$$

....

$$x(t_n) - y(t_n) + v_n = c_1 t_n + c_2 t_n^2$$

stoch. proc. noise trend function

....

v = noise with the expectation $E(v) = 0$

c_1, c_2 = unknown parameters

Site: z

$$z(t_1) - y(t_1) + v_1' = c_1' t_1 + c_2' t_1^2$$

$$z(t_2) - y(t_2) + v_2' = c_1' t_2 + c_2' t_2^2$$

....

$$z(t_n) - y(t_n) + v_n' = c_1' t_n + c_2' t_n^2$$

It is here anticipated that all our stochastic processes $x(t)$, $y(t)$ and $z(t)$ have a common signal $s(t)$ which is independent of the location of the site. Clearly, $s(t)$ represents the variations of type regressions and transgressions of the shorelines. We minimize the sum of $v_i v_i$ by selecting the least squares estimate of c_1 and c_2. Then we minimize the sum of $v_i'v_i'$ and obtain the least squares estimate of c_1' and c_2'. Having only three sites in our study we obtain for the joint solution after due elimination of s(t)

$$(f_1 + f_2)\, s^2 = vv + v'v'$$

where s^2 = variance in joint solution (pooled variance)

f_1 = degrees of freedom in the first sum of squares

f_2 = degrees of freedom in the second sum of squares

We can now repeat the procedure with (x_t) and (z_t) as anticipated optimal predictors. Our final choice of optimal stochastic process is the solution that gives least pooled variance.

In this way, we have defined a discriminator, that normally will give us a unique determination of the site that represents "the optimal eustatic process" in our set of observations.

Our discriminator is primarily develloped for studies of vertical movements, but might also be useful for studies of horizontal movements. However, the concept of eustatic factor is of course not directly useful for these movements.

The evaluation of the results from a study of this type will be discussed for some alternatives.

1. All sites have equal pooled variance. If the chosen sites have a global coverage, then they all represent the eustatic factor in the uplift process (or subsidence process). If the sites only have a lockal coverage, then the solution represents the local eustatic factor. It is obvious, that a local coverage will not make it possible to discriminate against any movement, that is common for all used sites. A common eustatic rise of sea level for all Scandinavian sites will be recorded as the local eustatic component in the uplift.

2. There is one site with least pooled variance. We have now found a unique

solution for the optimal eustatic process. If we only have a local coverage, then our solution will include any movements common for all used sites as an eustatic change. This means that we cannot exclude that "the optimal stochastic process" includes some small movements of different origin.

3. The records from the observation sites will always be contaminated by observation errors and the results will be influenced by this noise as well as gross errors.

4. The final solution is based on the hypothesis that a non-eustatic movement, which is not stochastic, will be of deterministic type and can be represented by a trend function of low order.

We have here chosen a n-th order regression analysis and we postulate that time is measured from present time to the deglaciation time.

We can of course also chose an exponential presentation of the uplift procedure.

$$x(t_i) - y(t_i) + v_i = (x(t_1) - y(t_1)) e^{-a(t_i - t_1)}$$

Here it should be natural to measure time from deglaciation to present time.

With the use of this type of discriminator we expect to avoid some of the arbitrariness, that can be traced in some of the earlier determinations of eustatic change. We have found it necessary to define a mathematical discriminator because the separation between tectonic and isostatic movements in the crust has been made with the use of a previous determination of the eustatic factor.

It should be noted that the eustatic factor found in this type of approach has a local character. It represents the uplift of the global sea level when anticipating that there is an observation site inside our set of stations, where the uplift of the crust is negligible. When subtracting the "optimal eustatic process" from the remaining observation sets, then we expect to find a smooth curve which only represents the crustal movements. We will not know a priori if the "crustal movements" are of tectonic type or isostatic type. In a more precise analysis, we have further to separate between elastic, viscoelastic and viscous movements. Special care must be used when handling uplift data from the last section of the deglaciation period. If there is a rapid deglaciation, then a fast geoid subsidence will create great differences between the true uplift of the crust and the apparent uplift. This means that observations from sites in the neighbourhood of a glaciated area will have recorded apparent uplifts, which look far too high during the last section of the deglaciation period.

We note that our type of solution is invariant with respect to the actual signal in the stochastic processes. We can have a stationary stochastic process or a non-stationary process. There is no need for an identification of an eventual covariance function.

The hypothesis of a common signal $s(t)$ for all the stochastic processes means that we cannot extend the solution unlimited. The geophysical parameters for the test area must be compatible. For example, the tide effect should be of the same magnitude all over in the test area.

It is obvious that this type of analysis makes no direct determination of the signal $s(t)$. If

$$s(t) = s(t + \tau) = s(\tau)$$

then the process is said to be weakly stationary. (So called covariance stationarity.) The process is <u>ergodic</u> if the covariance function, defined by the expectation

$$E\{s(t)s(t + \tau)\} = q(t)$$

can be determind from the observations.

The optimal prediction of the signal is then according to Wiener-Hopf

$$x = q_{xy} Q_{yy}^{-1} y$$

where

x = optimal predictor

Q_{yy} = covariance matrix of the observations y

q_{xy} = crossvariance vector between prediction and observations.

Walcott (1975) concluded that there is no unique solution to the eustacy problem. The solution system will normally be singular.

The invariance we have obtained is linked with our definition of the problem. Below follows a study of the eustatic change based on the records by Tooley (1974) and Mörner (1976).

4. THE TECTONIC FACTOR OF THE UPLIFT IN FENNOSCANDIAN GLACIATED AREA.

Several authors have indicated that the present Scandinavian uplift might be composed by two independent components. There is a primary movement of tectonic type, which has been active since the Precambrian. Then there is a glacioisostatic movement superimposed on this primary movement. (Schwinner, 1928; Heiskanen, 1939; Sauramo, 1939; Daly, 1940; Artyushkov and Mescherikov, 1969; Mörner, 1976.)

Most of the studies have no direct proofs for the existence of the tectonic movements in Fennoscandia. Only Mörner (1976) gives a more detailed analysis of field records as a support for his conclusions. We will include some further considerations.

Most of the studies mentioned here seem to accept the tectonic component as an explanation of the uplift profiles so far recorded. It is wellknown that there is a very rapid change in the uplift rate during the last section of the deglaciation period. At this time, we get a very complex pattern for the apparent uplift and the true uplift is masked by some interference from secondary effects, which were not considered in these earlier studies. We will give a very condensed presentation of mechanism when an ice-cap is melting down.

Definitions:

1. Apparent uplift: The uplift of the crust in relation to mean sealevel.
2. True uplift: The uplift of the crust in relation to the gravity center of the earth.
3. Geoid: The equipotential surface coinciding with mean sealevel.
4. Eustatic correction: The correction to mean sealevel obtained by dividing the volume change by the ocean surface.

YR. BP Sites:	1	2	3	4	5	6	7	8	9
7700	31.3	25.8	17.7	5.2	.4	-4.7	-9.7	-14.8	-4.3
7350	28.3	24.0	17.5	7.2	2.8	-2.0	-6.6	-11.2	-3.1
7100	25.6	23.6	16.6	7.0	3.1	-1.4	-5.6	-10.0	-3.0
6700	24.0	20.6	15.5	7.7	4.0	.0	-3.7	-7.7	-2.3
5825	20.8	18.0	14.0	7.7	4.6	1.5	-1.6	-4.8	-1.2
5200	17.2	15.0	11.8	6.8	4.0	1.3	-1.4	-4.2	-1.1
4100	13.8	12.0	9.8	5.6	3.5	1.6	-.3	-2.3	0.0
2600	9.0	8.0	6.4	3.8	2.8	1.7	.7	-.6	+0.9
0	0	0	0	0	0	0	0	0	0

The uplift records from the eight first sites are found in the graph by Mörner (1976). The sites have numbers which increase in the direction south. All sites are found in western Sweden.

The site number 9 has been obtained by interpolation from the figures by Tooley (1974).

We give below an example of the analysis made for each induvidual site. This example is valid for a second order fit. A similar study is made for a third order fit.

The conclusion from this study is that the eustatic change is not well defined from any of the Swedish sites. The standard deviation of any eustatic curve is at least 10 % of the predicted quantity. If we consider the eustatic change known without error, then the isostatic-tectonic crustal movement cannot be determined without addidtional errors of considerable magnitude when using available records.

In our determination of the optimal estimator of the eustatic change, we obtained the following variances from a selection of eight Swedish sites and one English site. (No 9 is the English site.)

V A R I A N C E S

Site	1	2	3	4	5	6	7	8	9
1		4.7	6.0	18.4	20.1	20.9	22.8	24.8	6.9
2	4.7		1.9	10.3	10.8	11.7	13.0	14.5	2.4
3	6.0	1.9		4.1	4.7	5.2	6.3	7.3	0.2
4	18.4	10.3	4.1		0.1	0.2	0.5	0.6	3.3
5	20.1	10.8	4.7	0.1		0.0	0.2	0.4	3.8
6	20.9	11.7	5.2	0.2	0.0		0.1	0.2	4.4
7	22.8	13.0	6.3	0.5	0.2	0.1		0.1	5.3
8	24.8	14.5	7.3	0.6	0.4	0.2	0.1		6.3
9	6.9	2.4	0.2	3.3	3.8	4.4	5.3	6.3	
Sum	124.6	69.3	35.7	37.5	40.1	47.1	48.3	54.2	32.6

We find that the site 9 has the least variance and represents the optimal estimate of the eustatic change. Site 3 has the smallest variance among the Swedish sites.

We conclude:

When only using the Swedish sites, then the optimal estimator is found as the stochastic process from site 3.

When making a joint solution for Swedish and English observations, then the optimal estimator is found from the site 9 (English observations). Mörner (1976) made a determination of the eustatic change which was most closely correlated with the records from site 7.

The determination of the eustatic change is rather unstable and any estimate of tectonic uplift will be dependent of the final choice. (This is specially important for the deglaciation phase.)

Example of a computation: Analysis with site 7 as reference. This is not the best site according to our investigations. However, the records coincide very well with Mörner´s eustatic curve.

Site	X_1	X_2	V^TV	s	s_{x_1}	s_{x_2}
1	1.231	0.4776	22.84	±2.14	±0.72	±0.107
2	1.069	0.4253	12.97	1.61	0.54	0.080
3	1.054	0.3001	6.27	1.11	0.33	0.055
4	0.8159	0.1404	0.45	0.30	0.10	0.015
5	0.4760	0.1063	0.17	0.19	0.06	0.009
6	0.2441	0.0505	0.08	0.12	0.04	0.006
7	....	....	....	....	....	....
8	-0.3258	-0.0415	0.06	0.11	0.04	0.005

$V^TV = 42.85$ (with the Tooley record included 48.10)

Observation equations: (h is observation value and v residual.)

$$tx_1 + t^2x_2 = h + v$$

Normal equations:

$$293.1531\ x_1 + 1937.0232\ x_2 = \Sigma\ th$$

$$1937.0232\ x_1 + 13200.7374\ x_2 = \Sigma\ t^2h$$

Covariance matrix:

$0.1121\ s^2$	$-0.01645\ s^2$
$0.01645 s^2$	$0.002489\ s^2$

Degrees of freedom for each site:5

Only the records from the sites 5, 6 and 8 have acceptable standard deviations in combination with site 7. (The third order solution reduced the standard deviations considerably for the three first sites.)

Modern theories of the uplift mechanism make it questionable if the eustatic change can be considered an invariant quantity that can be determined in a meaningful way from local observations. (See Farrel and Clark 1976.) A strict analysis should require that we make an integration of the change of the ocean volume for a defined time span and then we obtain

$$E = V/A_W$$

where A_W is the ocean surface.

Our analysis gives a discriminator that defines the "optimal eustatic process" for the available set of observations. It is interesting to note that our study revealed that the observations in NW England are optimal in a comparison with the Scandinavian records. This could be expected from the known uplift history in Fennoscandia.

The numerical results from our study are somewhat remarkable. A study of Farrell-Clark revealed that there is a zero-isoline, for the global uplift of sealevel above the crust, intersecting the north-west coast of England when using a viscoelastic solution with a relaxation time of 1000 years. Computations for a longer relaxation time are not yet available. The coincidence might be accidental but deserves further attention. The findings might be a support for the Farrell-Clark relaxation model.

We refrain from making a test of the significance of our solution. We cannot use a standard F-test because our sums of squares are not stochastically independent. However, there is little doubt that there is no significant difference between the determinations of the eustatic change from sites 3, 4, 5 and 9. If the Swedish sites had basic crustal movements common with the English site, then these movements remain undetected.

4.1 THE TECTONIC FACTOR IN NW-ENGLAND.

Mörner (1976) analysed the records by Tooley (1974) and concluded: "The isostatic curve decreases continuously from 8,000 to 5,000 B.P., after which it remains at 0.4 mm uplift per year."

We found the following figures from the graph by Mörner:

Yr BP	Uplift BP (isostatic-tectonic)
8000	7.9 m
7000	2.8
6000	0.6
5000	0.4
4000	0.4
3000	0.4
2000	0.4
1000	0.4
0	0.4

The author has not explicitely expressed his opinion on the constant uplift of 0.4 mm per year. We think that the only possible evaluation is a tectonic component because an isostatic uplift will gradually fade out and have zero as the limiting value.

We have analysed the figures and find that the recorded uplift before present has been constant for the last five thousands years. This is impossible and we note that the record for the time zero is misleading. We cannot have 0.4 m uplift BP for the time zero BP. Clearly, there is a translation of the height-scale, and all uplift values should be subtracted by 0.4 m. The recorded uplift per year is then zero, instead of 0.4 mm.

Conclusions: The figures from the Tooley record give no proof for the existence of a tectonic movement. The crustal movements indicated in the plot seem to be clearly isostatic and faded out 5000 years ago.

4.2 TECTONIC FACTOR IN WESTERN SWEDEN.

Mörner (1973 p. 9) concludes that "the Scandinavian uplift is complex, partly caused by glacio-isostatic uplift, which decreases continuously and is now more or less finished on the Swedish west coast, and by another force, probably an old tectonic force, which is responsible for almost whole of the present uplift on the Swedish west coast."

Mörner (1976 fig. 6) gives the isostatic curves for eight well defined local sites, which we have analysed in the beginning of chapter 4. It is interesting to note that all eight curves have a straight line as a limiting value when time (BP = before present) goes to zero. However, this is not in contradiction to the mechanism in an isostatic compensation. We refer to fig. 3 of this paper where an isostatic uplift model is displayed. The uplift in the centre is a straight line with some additional bending. The exentric point at a distance of 9^{0} can be described as a straight line for the time span 6000 BP - 0 BP.

We are not going to make any further studies of the records from the west-coast, but refer to the corresponding analysis for the records from the east-coast of Sweden. A similar study can be made with the west-coast records. But the result will be somewhat less conclusive.

5. TECTONIC FACTORS IN THE UPLIFT OF THE EAST-COAST OF SWEDEN.

There is a considerable uplift at the east-coast of Sweden which is wellknown from several studies. Excellent studies of the magnitude of the uplift have been made by de Geer (1924), Lidén (1913) (1938) and Mörner (1976).

Mörner (1976) compiled a very complete list of the known uplifts. He summarizes: "The graph demonstrates that the Swedish uplift is complex and composed of two factors; one glacio-isostatic factor that decreased continuously with time and distance from the perphery and died out some 2000-3000 years BP (Fig. 7), and one "tectonic" factor that has remained constant and is responsible for the present uplift."

Mörner´s evaluation of the Scandinavian uplift will be given some further consideration. The study will be made in two steps and we start with the data presented in the graph by Mörner.

The records: There are nine uplift profiles:

Time:	Record:
Present	Geodetic/levelling
3000 years BP	Geological
4000	"
5000	"
6000	"
7000	"
8000	"
9000	"

The geological records have been made with stratigraphic analysis (Lidén) and C^{14}-dating (Mörner).

Mörner (1977) gives in a graph the following uplifts in mm/year for the east-coast of Sweden.

U p l i f t i n mm/y e a r (Cf Appendix 2)

Site / Yr BP	+720 km	+420 km[x]	+190 km	0 km	-100 km	-160 km	-270 km
9000	26.3	31.1	20.0	13.2	9.6	7.9	3.0
8000	15.6	20.8	14.7	9.3	7.1	5.2	2.3
7000	13.5	17.5	12.4	7.4	5.7	3.7	1.9
6000	13.0	15.9	10.8	6.4	4.9	3.1	1.5
5000	12.4	14.3	9.8	5.4	4.3	2.9	1.4
4000	12.0	13.2	9.1	5.2	4.3	2.9	1.4
3000	11.6	12.5	8.6	4.9	4.3	2.9	1.4
0	11.4	12.0	8.3	4.9	4.3	2.9	1.4

x Ångermanland (from Lidén)

The given distances are positive north of Stockholm.

The uplifts are given in mm. Mörner has in private communication given the following definition of his uplift values:

Mörner: $$\text{uplift/year} = \frac{h_{BP}}{t_{BP} - t_P} \quad x)$$

t_{BP} = years BP

t_P = present

h_{BP} = total uplift between t_{BP} and t_P

x) Note that this definition is not invariant with respect to the choice of origin. The classical defintion uses the limiting value when the time difference goes to zero.

The geological records refer to seven sites with the following spacing:

No		Site:		Observer:
1	+	720 km	north	Mörner
2	+	420	Angermanland	Lidén
3	+	190	Gävle	Mörner
4		0	Stockholm	Mörner
5	-	100		Mörner
6	-	160		Mörner
7	-	270	south	Mörner

Lidén´s records and the geodetic records are given numerically. Mörner´s records are given graphically.

It is not known what type of discriminator Mörner used in his analysis, but we will start with a rather simple one.

Null hypothesis: The records from the 8000 BP profile and the "present" profile of uplifts per year have the same (isostatic?) origin.

Alternative hypothesis: The 8000 BP profile and the present profile have different origins.

Discriminator: The uplift rate per year is measured in each consecutive site from north to south. Increasing uplift is recorded + and decreasing uplift is recorded -. The following results are obtained

	1	2	3	4	5	6	7
Present	+	-	-	-	-	-	-
8000	+	-	-	-	-	-	-

The probability of a + or a - is considered to be 0.5 (independent outcomes). The outcome space for each site is

+ + - -

+ - + -

The probability of a prescribed outcome is $(0.5)^2 = 0.25$

The probability of obtaining 7 prescribed outcomes is $(0.25)^7 = 0.001$

We use a more conservative discriminator with the outcome space defined when the "present" outcomes are known ã priori. Then the probability of obtaining equal result for any of the 8000 records is of course 0.5. The probability of obtaining 7 equal results is then $(0.5)^7$ **= 0.008.**

Using this type of analysis we find that there is less than 1 % probability that the two profiles have different origin. We accept our null hypothesis of equal origin for the two events.

We could have included all remaining outcomes in our analysis and made it still more convincing.

C o r r e l a t i o n   a n a l y s i s:

We make a straight forward correlation analysis between the records from 9000 BP and the present uplift records from geodetic levelling. Then we find:

Geodetic levelling = x

9000 BP record = y

Standard deviations:

$$s_x = 4.16 \qquad s_y = 10.28$$

Correlation coefficient:

$$r_{xy} = 0.86$$

Clearly we have a very good correlation between the earliest uplift records and present geodetic uplifts. (A more advanced analysis will require correction for elastic, eustatic and geodial movements. The uplift rates should be computed as limiting values.)

Conclusions: Present uplift records from geodetic observations are very well correlated with earlier uplift records. We have not found anything that indicates that present geodetic uplift figures have a tectonic origin. If there is a tectonic movement then it must be very small

6. THE ELASTIC UPLIFT.

When using a **single** layer approach for the potential we obtain in the rigid earth case for the point loads t at the surface of a sphere

$$T_j = G\int_S (t/r)\, dS = G\int_S (t/r_j)\sum_{n=o}^{\infty} (r_o/r_j)^n\, P_n(\cos\theta)dS$$

where r is the distance between the fixed point (P_j) and the moving point, r_j geocentric distance of the fixed point, θ geocentric angle between the fixed and the moving point, T_j disturbance potential and S surface of integration.

For an elastic earth we have the corresponding formula

$$T_j = G\int_S (t/r_j)\sum_{n=o}^{\infty} (r_o/r_j)^n\, (1 + k_n - h_n)\, P_n(\cos\theta)dS$$

where k_n is the Love number for the potential change and h_n the Love number for the radial chance in an elastic approach.

Peltier (1974) computed the Love numbers for three interesting earth models. One of these models seems to correspond well to our present information about the earth and we give some selected data.

Earth model.

Mantle	0 - 300 km	10^{21} Poise
	300 - 3000	10^{22}
	3000 - center	inviscid

Earlier studies of the uplift problem were based upon the hypothesis of a homogeneous viscous mantle. The Peltier model uses a stratified viscoelastic mantle, described by so called s-spectra. The deglaciation period is dominated by the elastic movements and we can use the Farrell-Clark analysis for an estimate of the magnitude of the quantity.

A solution has to satisfy the following relation

$$\int_S h_W \rho_W \, dS = \int_S h_I \rho_I \, dS \qquad t_W = \rho_W h_W \text{ and } t_I = \rho_I h_I$$

where h_I is height of the ice sheet, ρ_I density of the ice, h_W additional height of sea level after melting of the ice and ρ_W density of water. The mass of the ice must be equal to the mass of the meltwater.

If the Love numbers are known, then we can compute the corresponding potentials

$$(T_W)_j = \frac{G}{r'_j} \int_S \rho_W h_W \sum_{n=o}^{\infty} (r_o/r'_j)^n (1 + k_n - h_n) P_n(\cos\theta) \, dS$$

$$(T_I)_j = \frac{G}{r''_j} \int_S \rho_I h_I \sum_{n=o}^{\infty} (r_o/r''_j)^n (1 + k_n - h_n) P_n(\cos\theta) \, dS$$

where T_I is the disturbance potential caused by the ice sheet and T_W is the distubance potential caused by the meltwater. For $r'_j = r_o$ we obtain the radial displacement for the meltwater ("crustal displacement")

$$g(h_W)_j = \frac{G}{r'_j} \int_S \rho_W h_W \sum_{n=o}^{\infty} h_n P_n(\cos\theta) \, dS + \text{const}$$

where g is gravity. The corresponding integral equation is valid for the ice. We obtain a final integral equation for the determination of the change in sea level from the two primary integral equations. The original mass-condition must be included.[x)]

Farrell and Clark (1976) computed the vertical displacements for a global model covered with ice according to known ice distribution in the Laurentide and Fennoscandian ice sheets. The results for the Baltic are given below.

[x)] A solution can only be obtained for known heights of the ice.

Elastic movements of the Baltic.

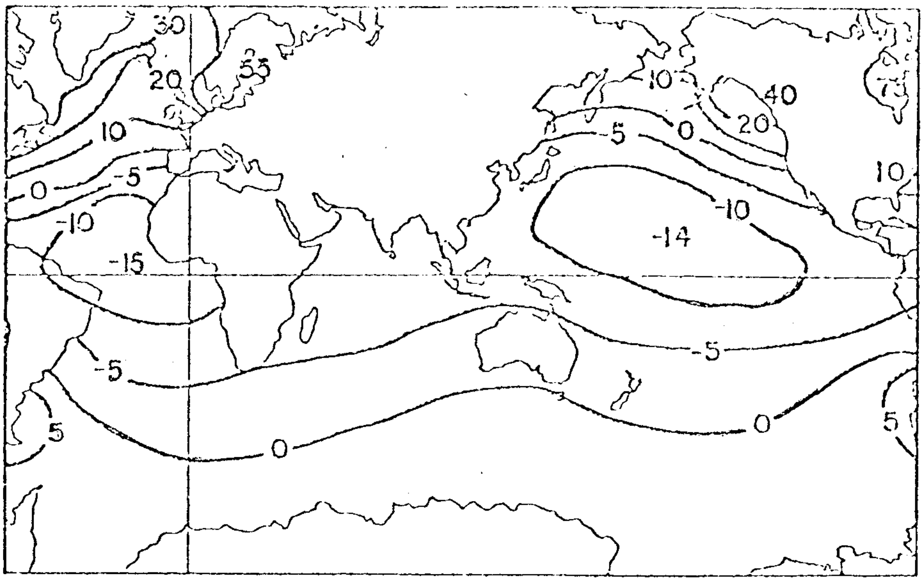
The melting of 1 metre of ice in the entire ice sheets of Fennoscandian and Laurentide origin gives the following elastic movements in the solution of Farrell and Clark.

E = 0.8 m	Global eustatic change. This is the expected apparent subsidence (crust) in a traditional solution.
+0.55 E	True crustal upheaval of the Baltic.
-0.5 E	Uplift of the geoid above the crust. We note negative sign which means that the geoid subsides relative the crust.
0.55E+(-0.5E)=0.05E	Apparent uplift of the crust. (Alternatively, apparent subsidence of the geoid.)
-0.05E+E = 0.95 E	Overestimate of the apparent uplift of the crust in the traditional type of solution. (Valid for the deglaciation phase.)

A solution is also made for the case when the entire Laurentide and Fennoscandian ice sheets melt instantaneously (but other ice sheets are held constant). The eustatic change is then E = 82 metres. The uplift of the geoid in the Baltic is then -1.30 E. The corresponding figure for the Hudson Bay is -2.70 E.

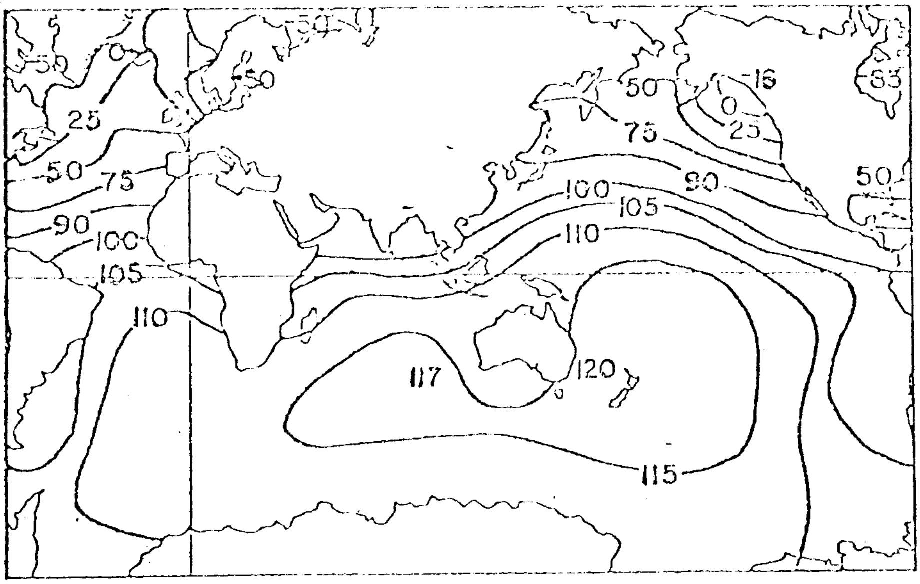
These elastic movements have not been considered in the traditional solutions. This means that the deglaciation phase has been misinterpreted.

(The figures given by Farrell-Clark must be considered preliminary and further investigations will follow.)



After: Farrell-Clark.

Fig. 1. Uplift of the crust in per cent of E (here 0.8 m) when 1 metre ice melts from the entire Fennoscandian- Laurentide ice sheets.



After: Farrell-Clark.

Fig. 2. Uplift of the geoid above the crust in per cent of E for the case described in fig. 1. The apparent uplift of the crust will be 0.05 E. The expected apparent subsidence of the crust was 1.oE in the traditional solution. This means an overestimate of 0.95 E.

7. THE VISCOELASTIC UPLIFT.

The theories of Peltier (1974) include a set of time dependent Love numbers for the computation of the "slow" uplift, which traditionally has been treated as viscous movement. See fig. 3.

Farrell and Clark (1976) have used the theories of Peltier for a realistic earth model which included the Laurentide and the Fennoscandian ice sheets. They computed the sum of the elastic uplift and the viscoelastic uplift for the following 1000 years when 1 m of ice has melted from the Fennoscandian and Laurentide ice sheets. See fig. 4.

There is a practical problem when using these advanced theoretical models. The Peltier solution operates with four Love numbers for each Legendre polynomial. Two of them are time dependent and two of them are s-dependent. There will of course be great difficulties in finding appropriate figures for each polynomial in a Legendre expansion. It can also be questioned if a spherical stratification is sufficient to give meaningful improvements. Seismic velocity records indicate that lateral differences in geological parameters can be rather large.

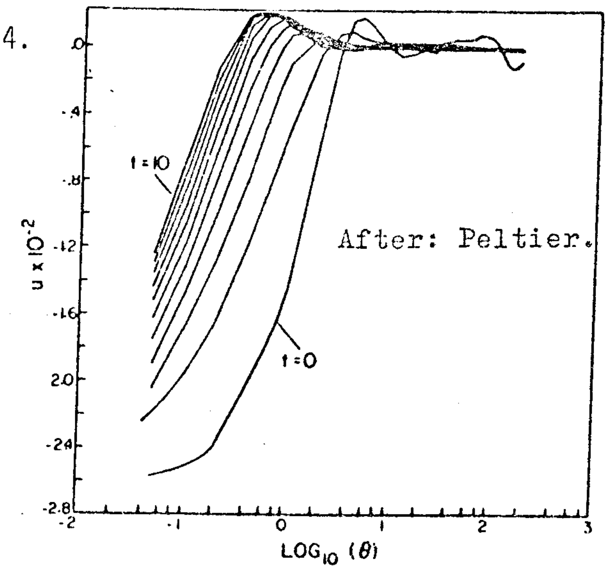


Fig. 3. Viscous part of the Green function for radial displacement. Time slices are at 1000-year intervals. Note the complex deformation of the peripheral bulge as it migrates inward.

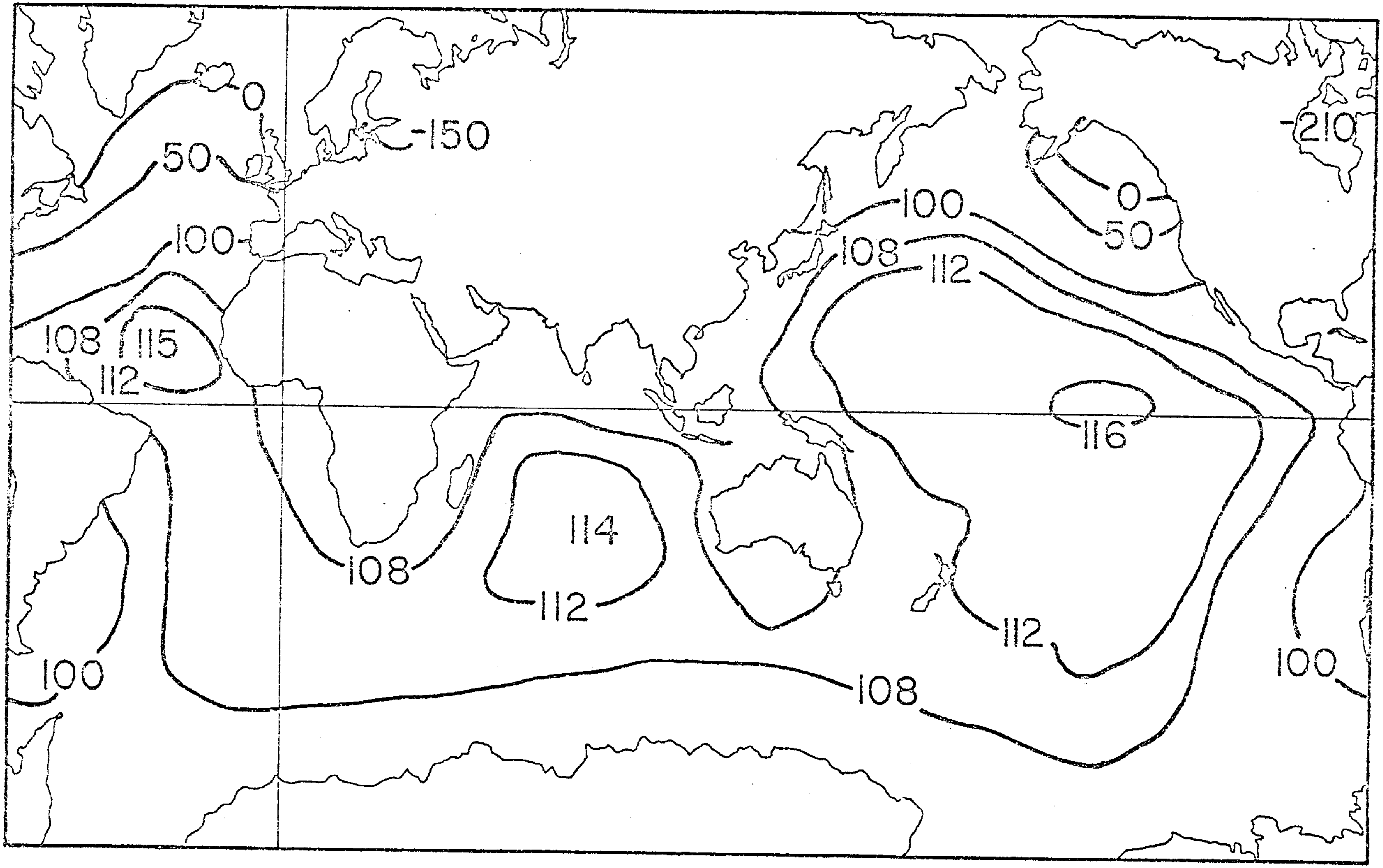


Fig. 4. Total change in sea level in per cent of E = 0.08 m after melting 1 metre from all ice masses and allowing the earth to relax for 1000 years. After Farrell-Clark. Movements of the crust are of the same magnitude.

8. THE VISCOUS UPLIFT.

Most studies of the uplift problem are based upon the hypothesis on earth that can be considered to be a homogeneous Newtonian viscous fluid. Modern studies of this type have been presented by Scheidegger (1957, 1963) and Mc Connel (1963, 1965, 1968).

There was only one fundamental free parameter in most of the studies, the Newtonian viscosity and they were normally only valid for a plane earth. Haskell (1935, 1936, 1937) also included the study of P and S wave velocities.

It was generally believed that the vicosity of the lower mantle was higher than the viscosity of the upper mantle. (See also the interesting papers of Munk and Mac Donald (1960) and Mac Donald (1966).) A study of the "equatorial bulge" by Mac Donald indicated that the present value corresponds to the value it had 9.5×10^6 years ago. This means that the viscosity should be $7.9\ 10^{25}$ poise. Goldreich and Toomre (1969) questioned the technique used by Mac Donald and showed that it was a result of a bias in the present spherical harmonic description of the earth. The gravitational energy in C_2 is overemphazized by the usual computation technique. They reached a value of 6×10^{24} poise. Mc Connell (1971) obtained the value of 6×10^{21}.

Niskanen made careful studies of the Fennoscandian uplift with the use of a theoretical model that consisted of a homogeneous viscous ball (without an elastic crust).

The hypothesis of isostacy has often been used for an elementary explanation of the uplift mechanism of the crust. Airy and Heiskanen used a simple model of the earth where the visible mountains had roots in the homogeneous viscous mantle. We use the same isostatic model for an ice sheet above the crust. Then we obtain the simple relation which should be valid when there is an isostatic compensation for the ice load

$$(h - x)\rho_1 = (\rho_3 - \rho_2)x + (\rho_2 - \rho_1)x$$

ρ_1 = density of the ice

ρ_2 = density of the crust

ρ_3 = density of the mantle

x = depth of the root

h = height of the ice sheet

When there is a full compensation of the ice load, then the crust will show subsidence of depth x, and we have the relation

$$x = h\,\rho_1/\rho_3$$

It is supposed that the crust is sufficient elastic or has sufficient with faults to permit a full isostatic compensation. If the ice and the mantle once were in a state of equilibrium, then the quantity x represents the total uplift to be expected. One can make a tentative determination of the old uplift and then compute the uplift to be expected ("remaining uplift"). The uplift rate can also be determined. Most studies seem to indicate that the viscosity of the mantle is in the order of 10^{21} Poise.

9. UPLIFT FROM A VISCOUS MANTLE WITH AN ELASTIC CRUST.

A homogeneous viscous mantle with an elastic crust has been used in a number of theoretical models studied by Leif Svensson. (Swedish geodynamical committee.) This type of approach has the advantage of having an acceptable complexity and is closer to reality than the studies that ignore the external crust. The study is mainly useful for a description of the uplift mechanism after the deglaciation has been completed. The fast elastic movements during the deglaciation period has to be computed separately.

We will here include one of the model studied by Svensson.

The theoretical model:

The earth has a viscous mantle with an elastic crust. All variations are so small, that a linearized Navier-Stokes' equation is acceptable for the description of the movements. The mantle is an unelastic fluid of high viscosity.

The crust and the mantle are homogeneous.

We note that this model is neglecting the elastic properties of the mantle. The Farrell-Clark model should be consulted for the elastic movements of the mantle. Their model neglects the elastic properties of the crust.

The following results were obtained for a model with a circular ice load. (Unpublished paper.) Units: MKS.

Radius of the ice load 9^o.
Height of the ice sheet 2000 m.
Density water 1000
Density crust 2700
Density mantle 3300
Density ice 900
Radius of the earth $6.37 \cdot 10^6$
Radius of the mantle $6.335 \cdot 10^6$
Transversal wave velocity in the crust 3000
Longitudinal wave velocity in the crust 5196

We start from isostatic equilibrium and postulate an instaneous melting of the ice. The crustal movements are listed below for the center of the anomaly field.

t	h	N	dg	h'
0	-530 m	-61.7 m	-68.3 mgal	68 m
2000	-405	-44.1	-52.4	52
4000	-314	-33.o	-40.8	39
6000	-245	-25.3	-31.9	30
8000	-192	-19.7	-25.o	23
10000	-150	-15.6	-19.6	19
12000	-117	-12.4	-15.2	14
14000	- 90.5	- 9.9	-11.7	11.6
16000	- 69.0	- 7.9	- 8.9	9.6
18000	- 51.7	- 6.3	- 6.7	7.7
20000	- 37.8	- 5.o	- 4.9	6.2
22000	- 26.6	- 4.o	- 3.4	5.o
24000	- 17.5	- 3.2	- 2.2	4.1
26000	- 10.2	- 2.5	- 1.3	3.2
28000	- 4.4	- 1.9	- 0.5	2.6
30000	- 0.2	- 1.5	- 0.1	2.o

Legend.

t = time in years after the melting

h = crustal subsidence (metres) = (remaining uplift)

N = geoidal subsidence (metres)

h' = dh/dt (t in thousand years) = (uplift rate)

dg = gravity disturbance (= dT/dr, T = disturbance potential)

This table is obtained with the use of an expansion in Legendre polynomials. We note that the crustal subsidence is strongly correlated with uplift rate, the geoidal subsidence and the gravity disturbance. (The gravity anomaly $\Delta g = -dT/dr - 2T/r_o$, r_o= radius of the earth.)

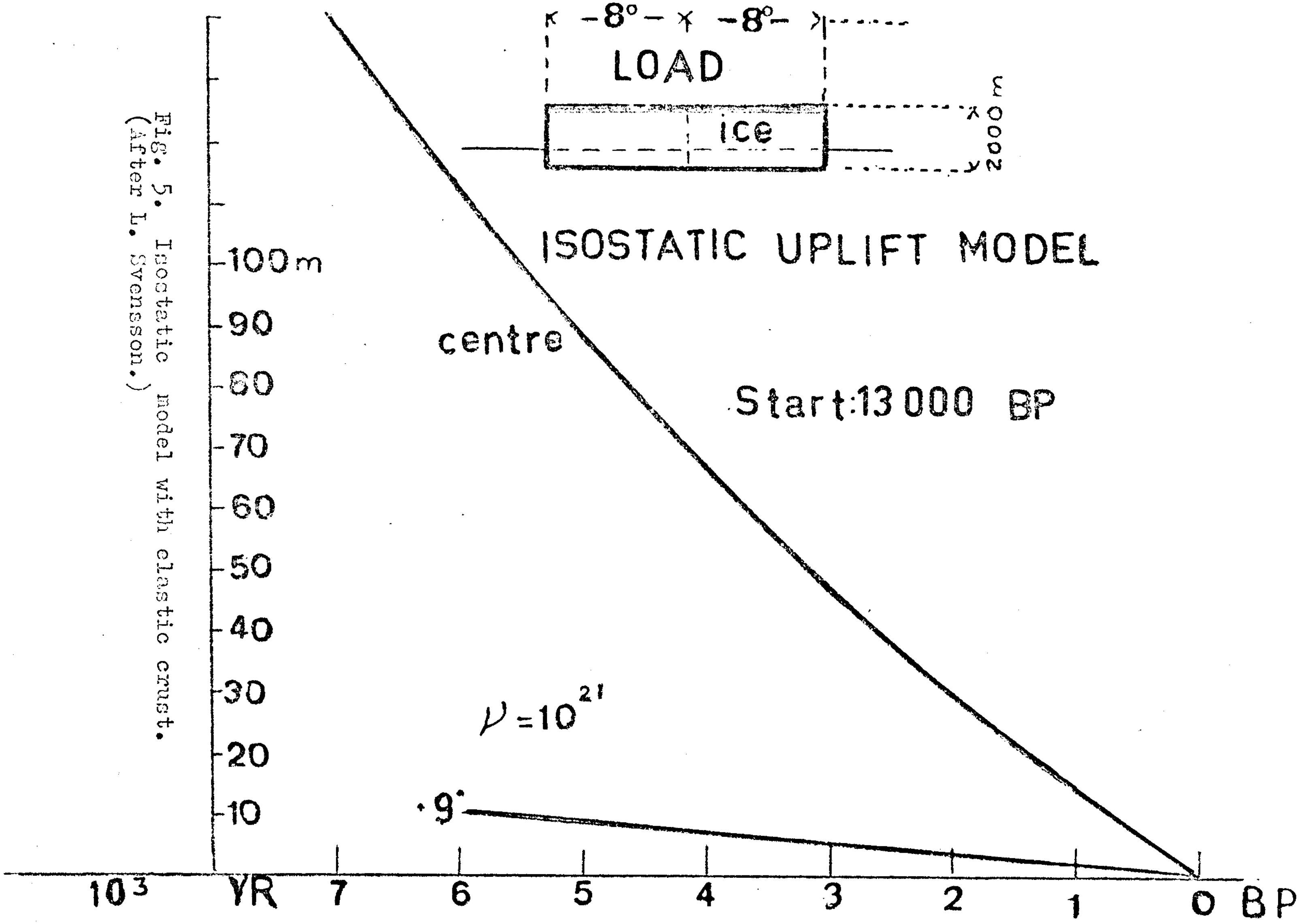


Fig. 5. Isostatic model with elastic crust.
(After L. Svensson.)

10. ANALYSIS OF THE FIELD RECORDS FROM THE CENTER OF THE FENNOSCANDIAN UPLIFT.

The present study is based upon the uplift records presented by Lidén (1938). The observations by Lidén have been made with the use of stratigraphic methods and the standard error of an observation is estimated to $\pm$ 1 dm. for heights which means that they are probably the most accurate now available. The original observations are referred to the river floor and Lidén has added a correction up to sea level. He has found that the depth of the river has varied between 2 and 5 metres. We estimate the error of this correction to $\pm$ 1 m. Time records have been given within 1 year and the error in time should be insignificant for this stratigraphic method.

We have also included a record of the eustatic change during the uplift time. This quantity is not so well known and we refer to a separate study of this quantity (see chapter 3.). The estimated standard error of the presented values of E is approximately $\pm$ 25 % of the recorded quantity, but we have no error analysis available. Anyway, the records will not be directly used in our analysis. They will only serve as a reference for our general discussion.

Crustal movements in northern Sweden (Ångermanland)

t	h	z	z'	E	h+E	h+E-z	h-z
9250	(281)	171.7	29.o	38	319	147.3	
8900	(232)	161.8	27.9	28	260	98.2	
8750	(218)	157.6	27.5	24	242	84.4	
8217	(194)	143.3	26.o	18.5	212.5	69.2	
7647	138.9	129.o	24.5	10	148.9	19.9	
6444	104.1	101.2	21.6	4.9	109.o	7.8	
5871	90.4	89.2	20.3	2.o	92.3	3.1	+1.2
5416	80.2	80.2	19.o	2.8	83.o	2.8	0.o
5238	76.2	76.8	18.7	1.7	77.9	1.1	-0.6
4057	54.4	55.6	16.8	0.4	54.8	-0.8	-1.2
3797	51.1	51.3	16.4	1.5	52.6	1.3	-0.2
3621	48.2	48.5	16.1	1.5	49.7	1.2	-0.3
3111	40.7	40.5	15.2	0.9	41.6	1.1	+0.2
2068	26.3	25.4	13.7	0.9	27.2	1.8	+0.9
1482	18.o	17.6	12.8	0.9	18.9	1.3	+0.4
1020	12.2	11.9	12.2	0.o	12.2	0.3	+0.3
0	0.o	0.o	11.o	0.o	0.o	0.o	0.o

Legend: (Unit for height 1 metre)

t = years before 1900 A.D.

h = apparent uplift (from Lidén) in metres

z = $185.2 \exp(-5416 + t)\, 0.000104738 - 105.o$

z' = dz/dt (t in thousand years)

E = eustatic change (Mörner, interpolation)

Values inside brackets are interpolated.

If we start with an analysis of the uplift records and the E-records, then we find that there is a strong influence of eustatic type in the time span 9250 - 6444 BP. The quantity E varies here between 38 m and 4.9 m. Farrell and Clark (1976) estimate the total change in E for the time span 18000 BP to present time to 82 m. The large eustatic changes have apparently faded out approximately 6000 BP. The time span 5416 - 0 has a peak value of E equal to 2.8 m and we conclude that this intervall is well suited for an analysis of viscous type. The previous time span of the deglaciation included very strong elastic movements combined with strong local movements of the geoid and are not suitable for a straight forward numerical analysis.

The time span 5416 - 0 includes an uplift of 80.2 m. We made a regression analysis using the following model.

$$z_o = z_n e^{a(t_o - t_n)}$$

$$\frac{dz_o}{dt} = 11 \text{ m} \quad \text{(from geodetic levelling)}$$

$$z_n - z_o = 80.2 \text{ m}$$

The analysis gave the following results:

$$z_o = 105.o \text{ m} \quad \text{(remaining uplift)}$$

$$a = 0.000104738$$

The values of z_i and z' are tabulated. The table also includes a computation of h-z and h + E - z. It is interesting to note that the standard deviation obtained from h - z is $\pm$ 0.7 m. The standard deviation from h + E - z is somewhat larger and there is no indication that the eustatic correction should give a smother result.

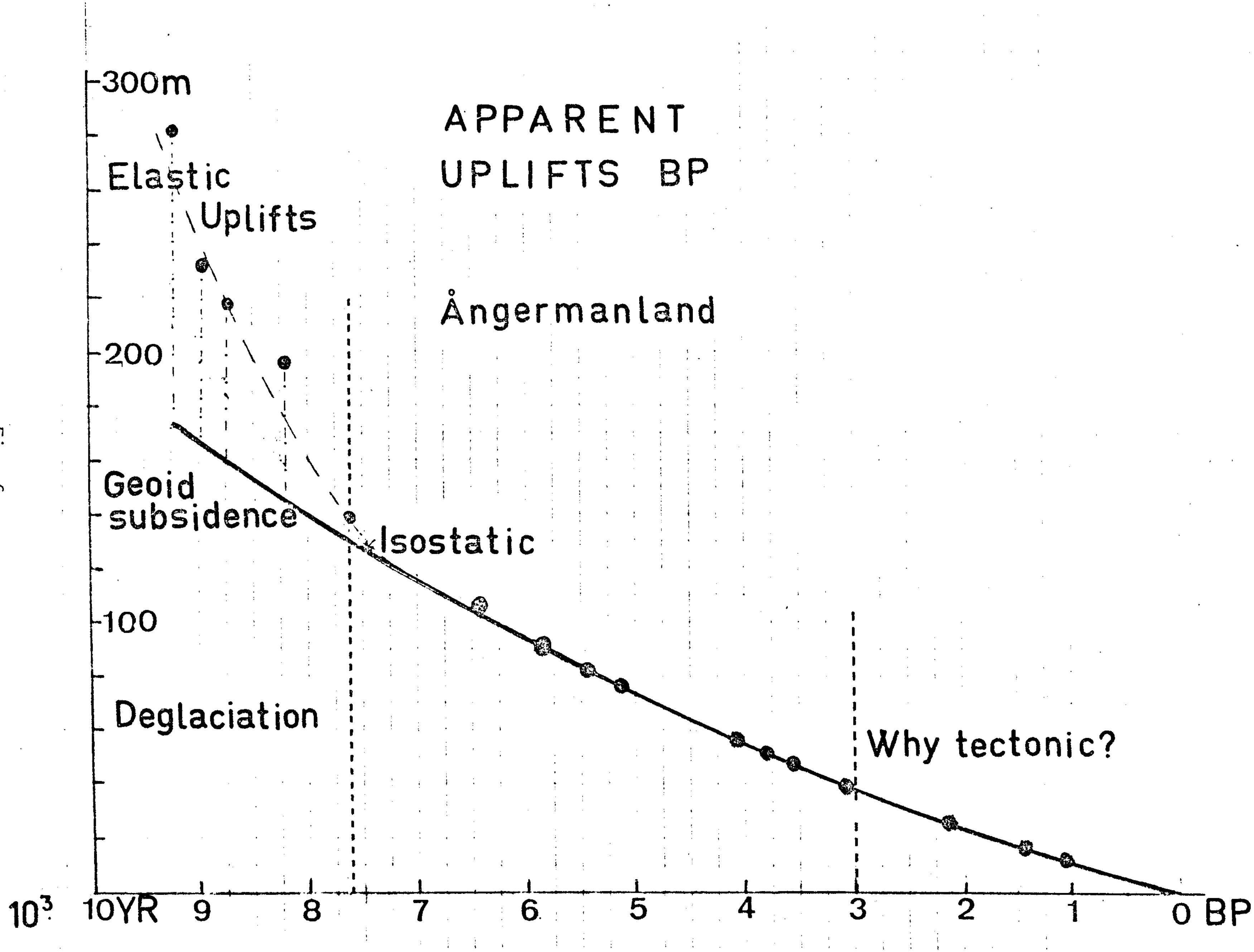


Fig. 6

Fig. 6. The graph shows the apparent uplifts from the Lidén records in Ångermanland. The elastic phase is obvious. A tentative line for the isostatic compensation is included.

Conclusions:

The analysis is based on an exponential function which is known to give a useful description of the uplift process if the initial uplift period is excluded. Our study has revealed that there is a good agreement between the selected analytical expression and the field records from Lidén. The standard deviation from the residuals is smaller than the anticipated standard error of the primary observations. All the recorded movements follow closely the pattern that is typical for isostatic compensation. We cannot find anything in these figures that gives a proof for the existence of a tectonic movement.

11. UPLIFT AND GEOPOTENTIALS.

Theoretical model studies indicate that there should be a correlation between the gravity anomaly and the vertical motion of the crust. Analyses of the gravity anomalies from the Scandinavian and the Hudson bay regions have indicated that there is a direct lack of correlation between the movements and the gravity anomalies. Jefferys (1940. 1970), Magnitsky and Kalashnikova (1970), Innes and Weston (1966). See however, also the studies of Gaposchkin and Lambeck (1971) and Kaula (1972).

The gravity field of the earth has often been studied in order to estimate density anomalies in the interior of the earth. It is well known that this "inverse problem" has no unique solution and therefore we have to be careful when we try to give the gravity anomalies a geophysical meaning. There is little doubt that the gravity anomalies reflect density variations, but the exact correlations are difficult to find. Lambeck (1976) concludes that most of the power in harmonics of degree ≥ 6 reflects heterogeneities in the first 300-400 km of the mantle. "The lowerdegree harmonics reflect conditions down to about 800-1000 km." It is also known that geopotential harmonic of degree ≥ 6 correlate with the surface topography. No correlation has been found between gravity anomalies and deep mantle seismic signals. Lateral temperature anomalies are supposed to be correlated with the gravity anomalies. Hide and Horai (1968) supposed that the gravity anomalies originated from the interface at the core mantle. Bott (1971) anticipated that the transition zone of the upper mantle was the origin of the anomalies. Anyway, the earth is not a rigid body and we can therefore expect a positive correlation between vertical movements in the earth and gravity anomalies. Kaula (1972) evaluated the gravity field of Gaposchkin and Lambeck (1971) as a flow in the asthenosphere and a corresponding response in the lithosphere. Kaula (1963) and Lambeck (1976) studied the degree variances of the geopotential. Lower spectra from the degree variances was used for extensive statistical correlation analysis. Lambeck concluded that the observed power spectrum of the earths gravity field can be interpreted as resulting from randomly distributed density anomalies of relatively small spatial coherence and occurring throughout the first 800-1000 km of the mantle. Lewis and Dorman (1970) indicate that there is an isostatic compensation which is significant down to a depth of 400 km.

The traditional method of analysing the uplift of formerly glaciated areas includes a determination of the "eustatic change" of the ocean sea-level, originated by the meltwater from the ice. It was anticipated that there was a uniform change in the ocean depth. This is a rather crude approach and refined earth model studies by Peltier (1974, 1976) and Farrell and Clark (1976) indicate that a viscoelastic earth model gives little support for earlier conclusions based on the mechanism anticipated in the determination of the eustatic changes. Any computation of uplifts from shoreline observations will be dependent on the method chosen for a determination of the eustatic change. It seems therefore important to use a study of the gravity field in order to find an independent estimate of the uplift forces now active in a selected area of the crust. We have already seen that the interpretation of the gravity data is rather intricate and has to be used with great care. We have to rule out the technique of statistical studies of power spectra because we are looking for "on site" analysis of a given area. The method we have chosen is selected in such a way that we expect to include maximum information about the selected geophysical parameters but unwanted information issuppressed. We use a type of "harmonic window".

The potential field of the earth is presented with the use of spherical harmonics.

$$V = (GM/r_j) \sum_{n=2}^{\infty} \sum_{m=o}^{n} (r_j/r_o)^n \, (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} \sin\phi$$

G is the gravitational constant, M is the mass, r_o is the radius of the earth and r_j is the geocentrical distance to the actual point. $P_{nm} \sin\phi$ are fully normalized assosiated Legendre polynomials. Furthermore, ϕ is latitude and λ is longitude.

These harmocics represent the total global gravity field. An isostatic subsidence in the Scandinavian geoid will have a very limited impact on high harmonics and on very low harmonics. This is obvious because the high harmonics only reflect density anomalies which originate from the upper mantle. Very low harmonics have a rather global nature and should therefore be avoided. Various "harmonic windows" have been contemplated and we present the results from one of the investigations.

The size of the "harmonic window" was selected in such a way that all available harmonics from $C_{10,0}$ - $C_{30,0}$ and $S_{10,0}$ - $S_{30,0}$ were included. This window should exclude meaningless low order harmonics, which only give a bias in the presentation of the glacio-isostatic gravity field. Furthermore, the statistical noise from the high harmonics should be properly suppressed.

The result of an analysis of this type is rather surprising. The total geopotential field of the earth is presented in fig. 7. We see here that the geoid (global sealevel) is 30 - 40 m above the international ellipsoid (approximately geoid with only C_{20}). There seems to be a good correlation between the Laurentide glaciated area and the present depression of the global geoid in Canada. However, in the Scandinavian area we have a negative correlation between the geoidal heights and the former glacio-isostatic subsidence. This reverse correlation has been discussed by several authors and Jefferys concluded that there must be something wrong with the gravity anomalies in Scandinavia or with the theory of isostacy. We will now look upon Scandinavia with the use of our selected harmonic window. Then we find a completely different picture, fig. 8. For detailed pictures of Scandinavia see fig. 9 and 10.

A well defined subsidence is now found in the local geoid of Scandinavia. This subsidence is very well correlated with the former Yoldia sea from the post glacial time. The present analysis indicates that this suppression amounts to 12 m in the Baltic. This means that the remaining isostatic uplift should be in the order of 100-200 m.

The final analysis of the gravity data is not yet completed and some small modifications can be expected. However, we find it obvious that there is a local supression in the Scandinavian geoid. This subsidence is hardly visible in the conventional global presentation of the geoid. It is masked in the general European-Atlantic highlands of the geoid. The subsidence is furthermore fairly well correlated with present uplift rates for Scandinavia.

This means that there seems to be a well documented isostatic uplift force in Scandinavia, and it seems natural to conclude that the main present uplift has an isostatic origin. We can still question if the uplift is glacio-isostatic or has some alternative explanation. A study of the uplift rates for the last 8000 years and the present uplift rates computed from repeated levelling indicates that there is an excellent correlation between present uplifts and the old uplifts in Sweden. This means that it seems to be justified to conclude that the glacio-isostatic uplift has not ceased in Scandinavia and that it is the main cause of the vertical movements in this area. Other types of movements are certainly also present, but their magnitude is probably very low. Small horizontal movements between blocks can be expected, but most earthquakes in this area are very small. (Magnitude 4 or smaller.) Nothing indicates that these small horizontal movements are sufficient to build up the large vertical movements now recorded. Note, for example, that the vertical movements in Scandinavia are of the same magnitude as the vertical movements in Thingvallier (Iceland) and S:t Andreas fault (USA).

Our study with the harmonic window has been verified in a complementary study of the Fennoscandinavian geoid using local terrestrial gravity data (40000 gravity observations). This study gave a geoidal subsidence of the same magnitude as the "harmonic window".

A special study has also been made with the use of the residual gravity anomalies obtained when subtracting the spherical harmonic (degrees 2-30) solution from the observed local free air anomalies. For further details see below.

12. LOCAL GRAVITY STUDIES FOR FENNOSCANDIA

The gravity disturbance represents the magnitude of the force vector for the disturbing potential. From the spherical harmonic solution we find

$$\frac{dT}{dr_j} = -\frac{GM}{r_j^2}\sum_{n=2}^{30}\sum_{m=o}^{n}(r_o/r_j)^n(n+1)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}\sin\phi$$

where $C_{20} = 0$. This quantity is plotted in fig 11. The plot is highly interesting and shows a great resemblance with the solution when using a harmonic window for the computation of a geoid. However, the isoline for -24 milligal follows approximately the zeroline for the geoid.

The free air gravity anomalies were computed from the spherical harmonic solution with the use of the formula

$$\Delta g_s = \frac{GM}{r_j^2}\sum_{n=2}^{30}\sum_{m=o}^{n}(r_o/r_j)^n(n-1)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}\sin\phi$$

A plot of this quantity is found in fig 12. The plot is almost a inverse picture of the plot of the gravity disturbance. The free air anomaly of gravity is basically an auxilliary quantity introduced in order to facili-ate the mathematical operations in a solution with terrestrial gravity data. We note that this quantity is obtained as a diffrence between gravity measured at the surface of the earth and the theoretical gravity from a reference earth. The theoretical gravity is measured at an auxilliary point in the vertical through the actual point and with a potential in the theoretical field that is the same as the potential for the actual point.

Terrestrial gravity anomalies have been computed with the use of the formula

$$\Delta g_t = g_{observed} - g_{theoretical}$$

A difference anomaly has been defined in the following way

$$\Delta g_{01} = \Delta g_t - \Delta g_s$$

These difference anomalies contain all available information above degree 30 in the spherical harmonic expansion.

The gravity anomalies have been measured at different altitudes and we make a reduction down to a common reference surface sphere by solving an integral equation of the following type

$$\Delta g = \frac{r_j^2 - r_o^2}{4\pi \; r_j} \iint_S \frac{\Delta g^{\cdot}}{r_{ji}^3} \, dS$$

$\Delta g^{\cdot}$ = reduced gravity anomaly

r_j = geocentric distance of the actual point

r_{ji} = distance between the actual point and the moving point on the sphere S

S = surface of integration (sphere)

For further details see Appendix.

When the gravity reduction has been completed, then the corresponding sea level undulations can be computed.

$$Ng = T = \frac{1}{4\pi \; r_j} \iint \Delta g^{\cdot} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos\theta) dS$$

where θ is the geocentric angle, T the disturbance potential, $P_n(\cos\theta)$ Legendre polynomial, g gravity and N sea level undulation (geoidal undulation).

We have completed this solution with the use of Dirac impulses at the internal sphere. (See Appendix.)

The number of terrestrial gravity anomaly data was in the order of 40000 for the Fennoscandian area. The solution was preceeded by forming integrated means for surfaces of the area 20' in latitude and 40" in longitude. A linear system with approximately 1900 unknowns was used in the final solution.

This type of solution contains all information in the spectrum which has the limits (approximate)

$$30 < \text{Spherical harmonic degree} < 500$$

A plot of the geoid is found in fig. 17.

13. EVALUATION OF THE GRAVITY SOLUTION.

Our present studies have been made with the use of satellite observations for the spherical harmonic solution and terrestrial gravity data for the refined structure of the gravity field.

The satellite solution is based upon the data in Goddard 7 and 8 and finally Grim 2.

The terrestrial gravity data have been generously supplied by the geodetic institutes in Danemark, Finland, Norway and Sweden. An important contribution with gravity data has also been made by the geological institute in Norway. We want to express our gratitude to the contributing institutes.

Our analysis has been made in several steps and we are here going to discuss each step individually.

The complete global solution.

The geoid is computed with the use of spherical harmonic expansion including the anomaly field up to degree 30.

We recognize some of the most interesting geoidal undulations.

Areas with strong positive geoid anomalies:

1. New Guinea (+ 81 m).
2. North Atlantic east of the continental rift. This "Atlantic platform" goes all the way to the Indian Ocean. All Europe and Africa are inclosed here. There is a maximum south of Africa and another maximum between England and Norway (+ 50 m).
3. South America.

Areas with strong negative anomalies:

1. Ceylon (- 113 m).
2. Canada
3. East of Mexico.
4. West of Mexico.
5. Antarctic.

One could expect that the areas which once have been glaciated should have strong negative anomalies. This is also the case in Canada, but not the case in Fennoscandia! No deglaciation has been recorded for India and still we find a strong negative anomaly. Some type of denundiation could be considered for the Indian ocean but the link is not obvious. So far we only know that isostatic equilibrium is not generally valid for the earth. One could expect that most of these low order anomalies have their origin deep in the mantle. Convection currents may be one of the contributing factors. Continental drift must also have an influence on the global anomaly field. Finally, we note that the choice of reference ellipsoid is also critical. All undulations are here presented with reference to the International Ellipsoid from 1967.

We are here restricting our study to the correlation between the glacio-isostatic uplift and the geoidal anomalies. Uplifts have been recorded after the Laurentide and Fennoscandian glaciations. The Laurentide glaciation has a strong correlation with present negative geoidal anomalies and will not be further analysed here. The Fennoscandian uplift area is situated in the Atlantic belt of positive geoidal anomalies. However, the size of the Fennoscandian glaciated area was of such a magnitude that most energy should be found in the spherical harmonic coefficients of order 15. We have therefore filtered out spherical harmonics of lower degree than 10 and obtain a new type of geoidal map from the satellite solution. This geoid includes only harmonics between 10 and 30.

Our study has been devoted to the problem of finding the residual gravity disturbances in the gravity field of Fennoscandia which can have their origin in the Pleistocene deglaciation (last deglaciation). and we have found a deep subsidence in the Baltic which is very well correlated with the formerly glaciated area. The zero isoline for the subsidence corresponds surprisingly well to the boundaries of the glaciated Fennoscandian area 10 000 years ago.

The peak value of the subsidence is -12 m. This peak value is found in the Baltic at a latitude of 59^{o}. We note that the peak value for the present uplift is found further north at the latitude 62.5^{o}. We know that the low harmonics included in the actual solution should be quite useful for the determination of the general size of the glaciated area. However, peak values require very high harmonics for a useful estimate. This means that we can only obtain a crude determination of the peak value from the spherical harmonic solution with satellite data.

We have therefore to look for the true peak value in the terrestrial solution which includes higher harmonics of degrees up to 500. Such a solution is found in fig. 17 (and fig. 21). We find here an additional subsidence of 9 metres at latitude $62.2^{o}$and longitude 20^{o}.This implies that the satellite solution and the terrestrial solution give a joint peak value in the subsidence of - 19 m . The location of the site coincides almost exactly with the maximum of the present uplift in Fennoscandia. (Ångermanland in Sweden.)

Conclusions:

Our two gravity solutions have fully confirmed the hypothesis that the present uplift in Fennoscandia should have an isostatic origin. The satellite solution confirms the general orientation of the site and the terrestrial solution confirms the location of the peak value.

A careful reader will note three competing peak values in the terrestrial solution, and we have to expect a multitude of local peak values when going up in harmonics. It seems justified to rule out the hypothesis of an isostatic origin for the three alternative peaks in the terrestrial solution, because they are very excentric with respect to the low order solution.

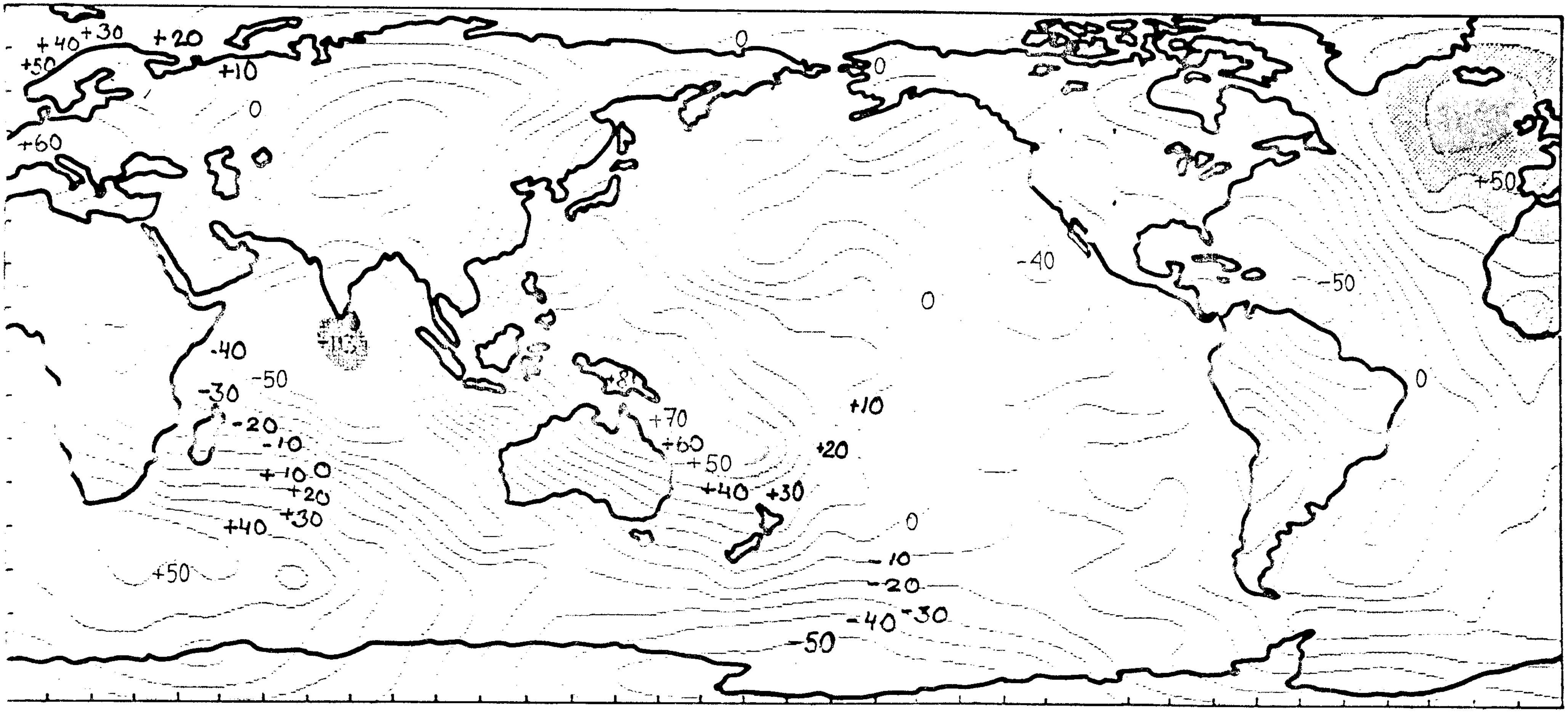
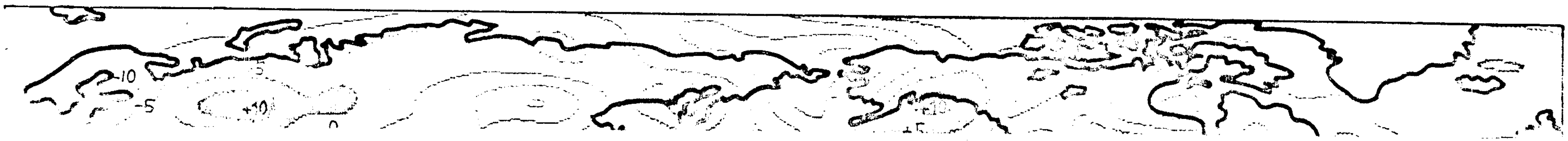
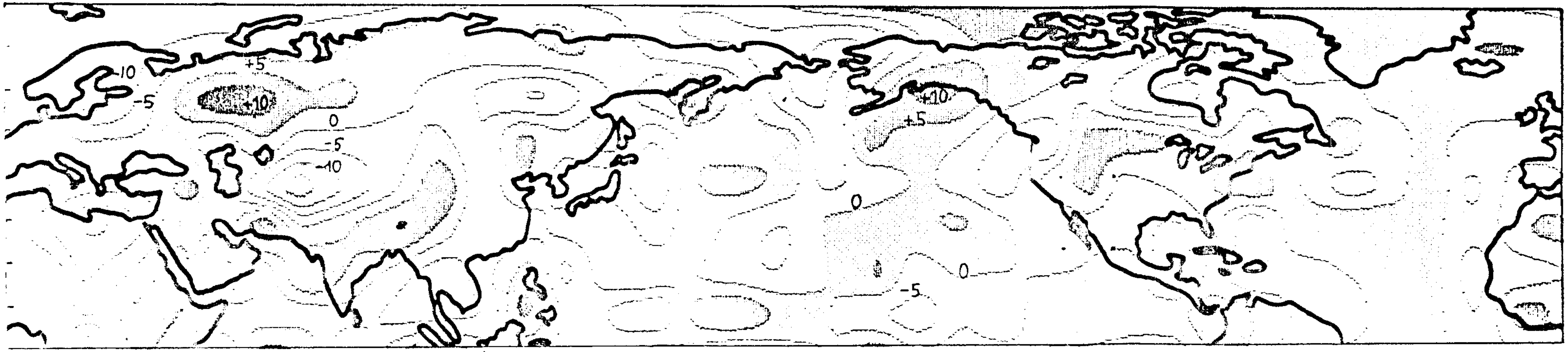


Fig. 7. Global map of the geoid. Reference ellipsoid IAG 1967
Data: Goddard Flight Center (Goddard 7 and 8) GRIM 2.
Spherical harmonics degrees: 2-30 (from Bjerhammar FoF 1/77)
Unit: 1 metre.





FoF 1/77

Fig. 8. Global map of the geoid when using a harmonic window with the spherical harmonic degrees: 10-30
Data: See fig. 7. (from Bjerhammar FoF 1/77)
Unit: 1 metre.

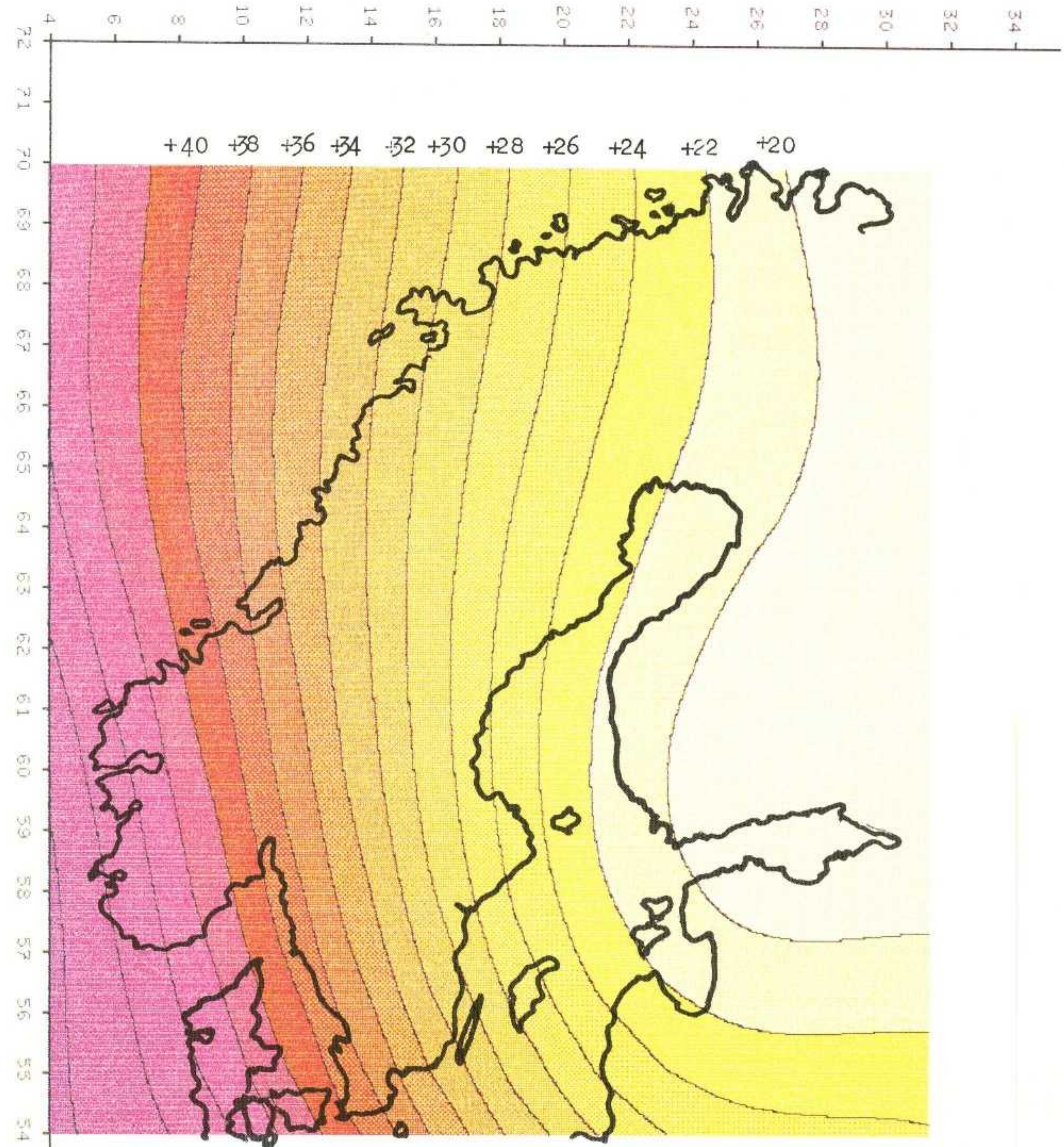
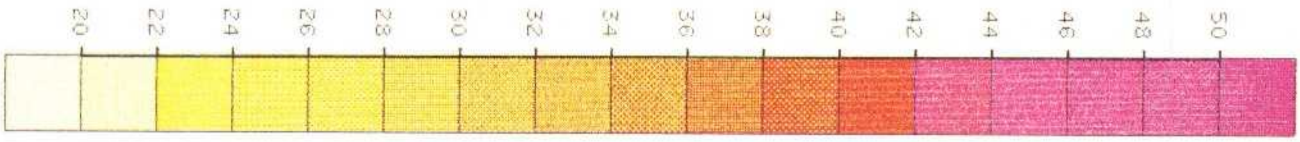


Fig. 9. Geoidal heights in Fennoscandia. Reference ellipsoid: IAG 1967.
Data: Satellite data from Goddard flight center. GRIM 2.
Spherical harmonic degrees: 2 - 30. Unit: 1 m.
Conclusions: Fennoscandia is highly elevated above the international reference ellipsoid. Negative correlation with the glacioisostatic subsidence. "Satellite geoid".



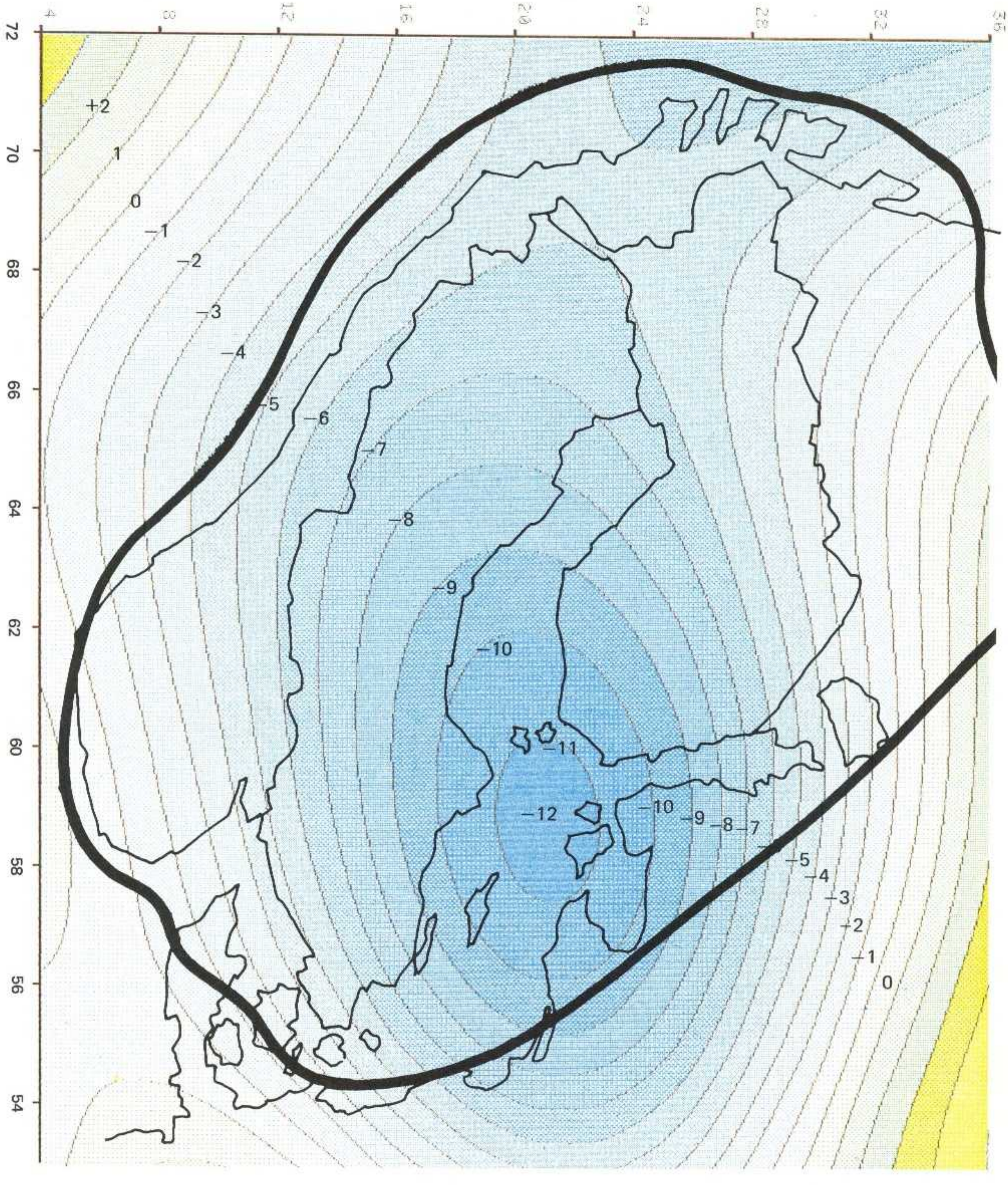


Fig. 10. Geoidal heights in Fennoscandia. Reference ellipsoid: IAG 1967.
Data: Satellite data from Goddard flight center. Grim 2.
Spherical harmonic degrees: 10-30. Unit: 1 m.
Conclusions: When using this harmonic window, then we obtain a geodial subsidence with boundaries very well correlated with the limits of the Fennoscandian glaciation approximately 10000 years ago.

"Difference geoid".

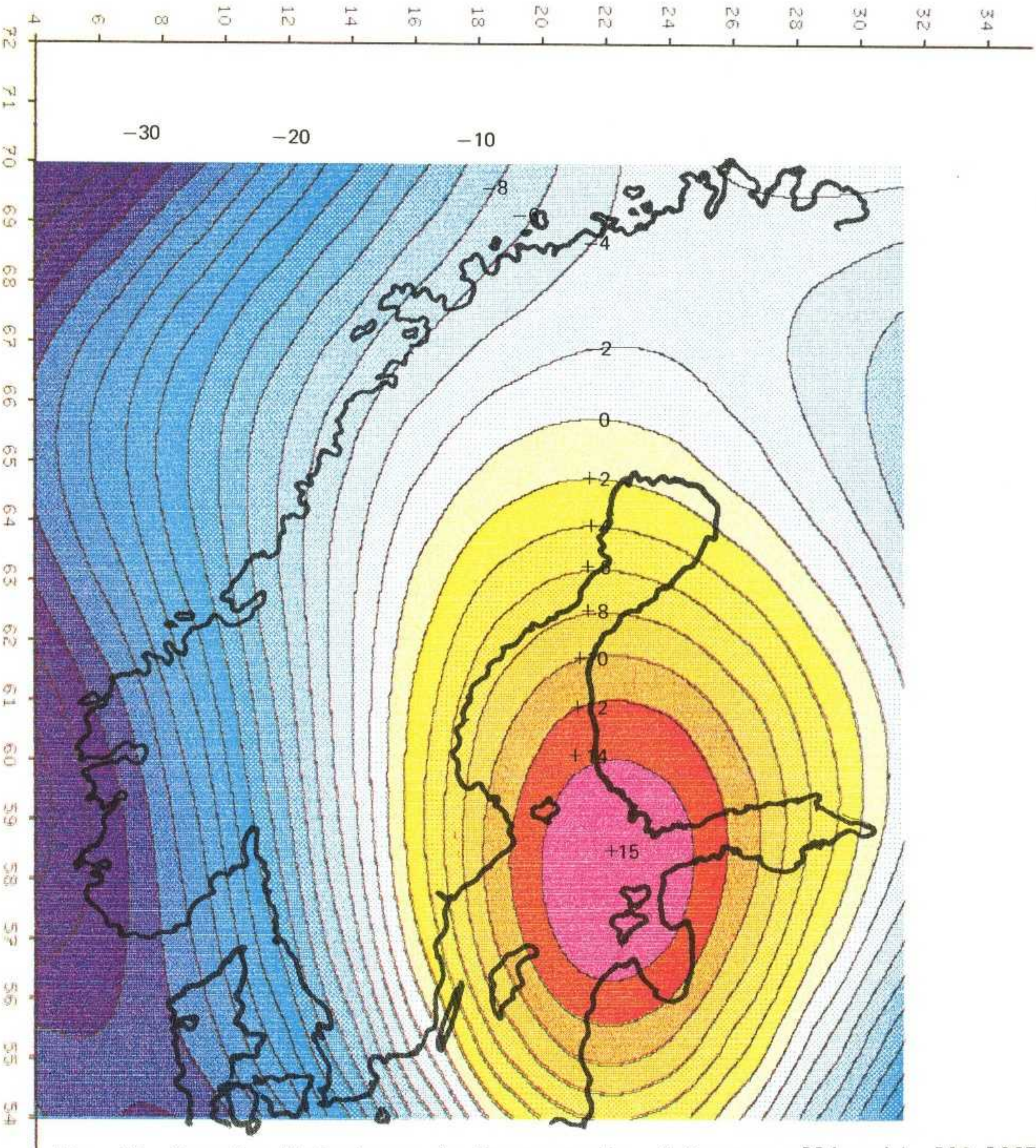
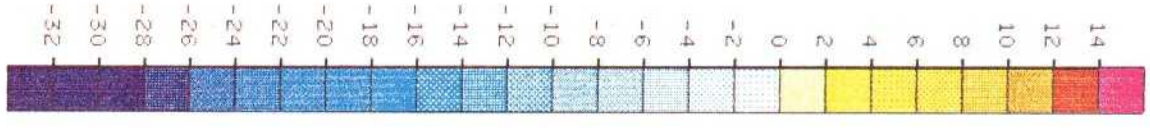


Fig. 11. Gravity disturbance in Fennoscandia. Reference ellipsoid IAG 1967.
Data: See fig. 9. Unit: 1 milligal. Mapped quantity: - (dT/dr).
Conclusions: A strong positive force anomaly is centered to the middle of the Baltic. Peak value: 14 milligal. "Satellite geoid solution".



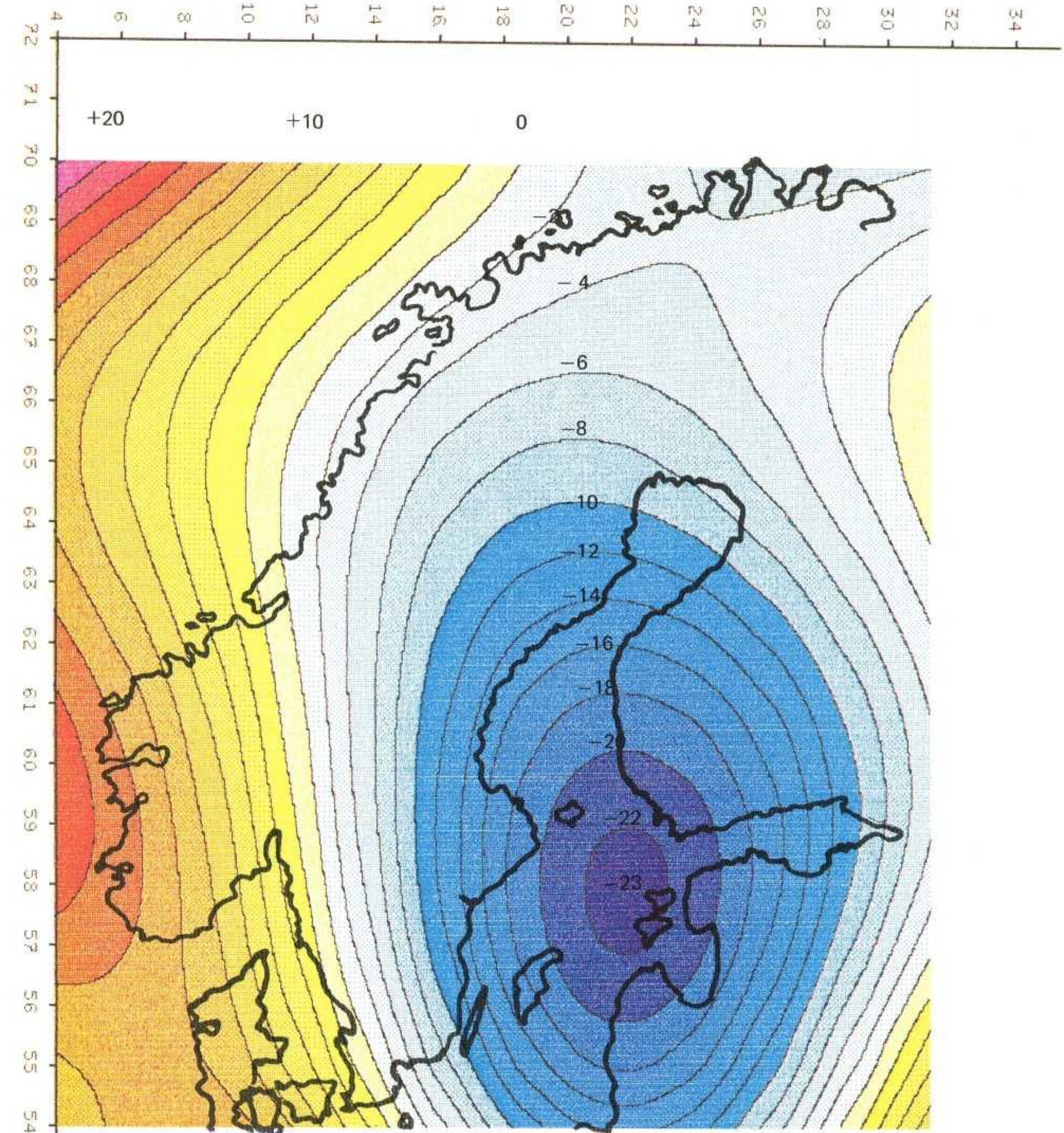
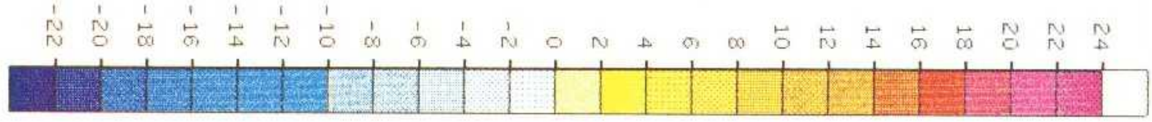


Fig. 12. Free air anomalies of gravity in Fennoscandia. Reference ellipsoid. IAG 1967. Data: See fig. 9. Unit 1 milligal. Conclusions: Fennoscandia is situated in the highlands of the global geoid. The gravity anomalies give therefore a distorted picture of the true force field. "Satellite geoid solution".



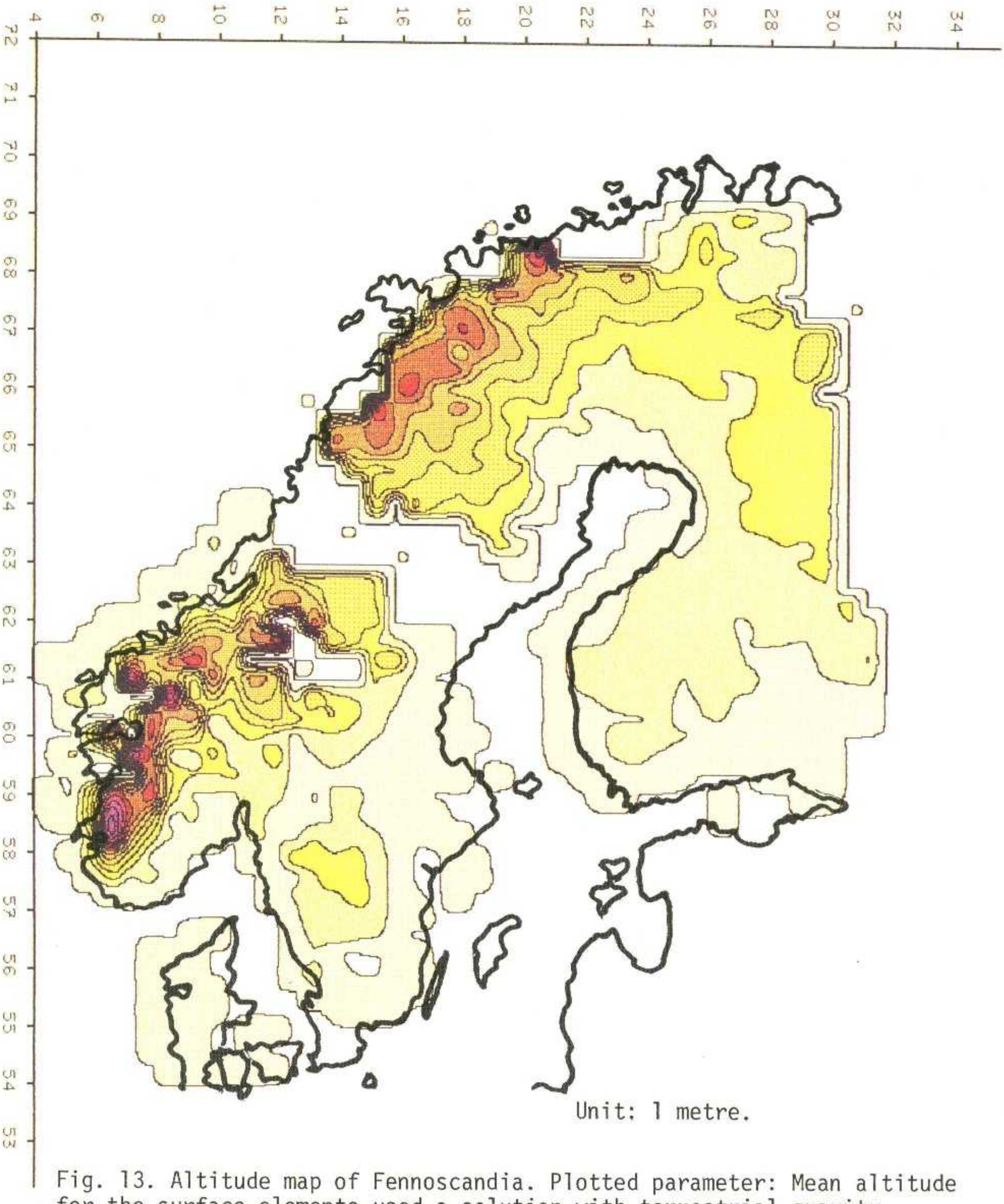
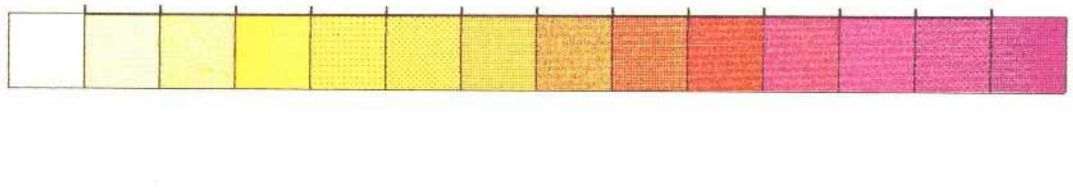


Fig. 13. Altitude map of Fennoscandia. Plotted parameter: Mean altitude for the surface elements used a solution with terrestrial gravity anomalies. Grid size: 20' latitude and 40' longitude. Surface elements without observations above sea level have no colour.



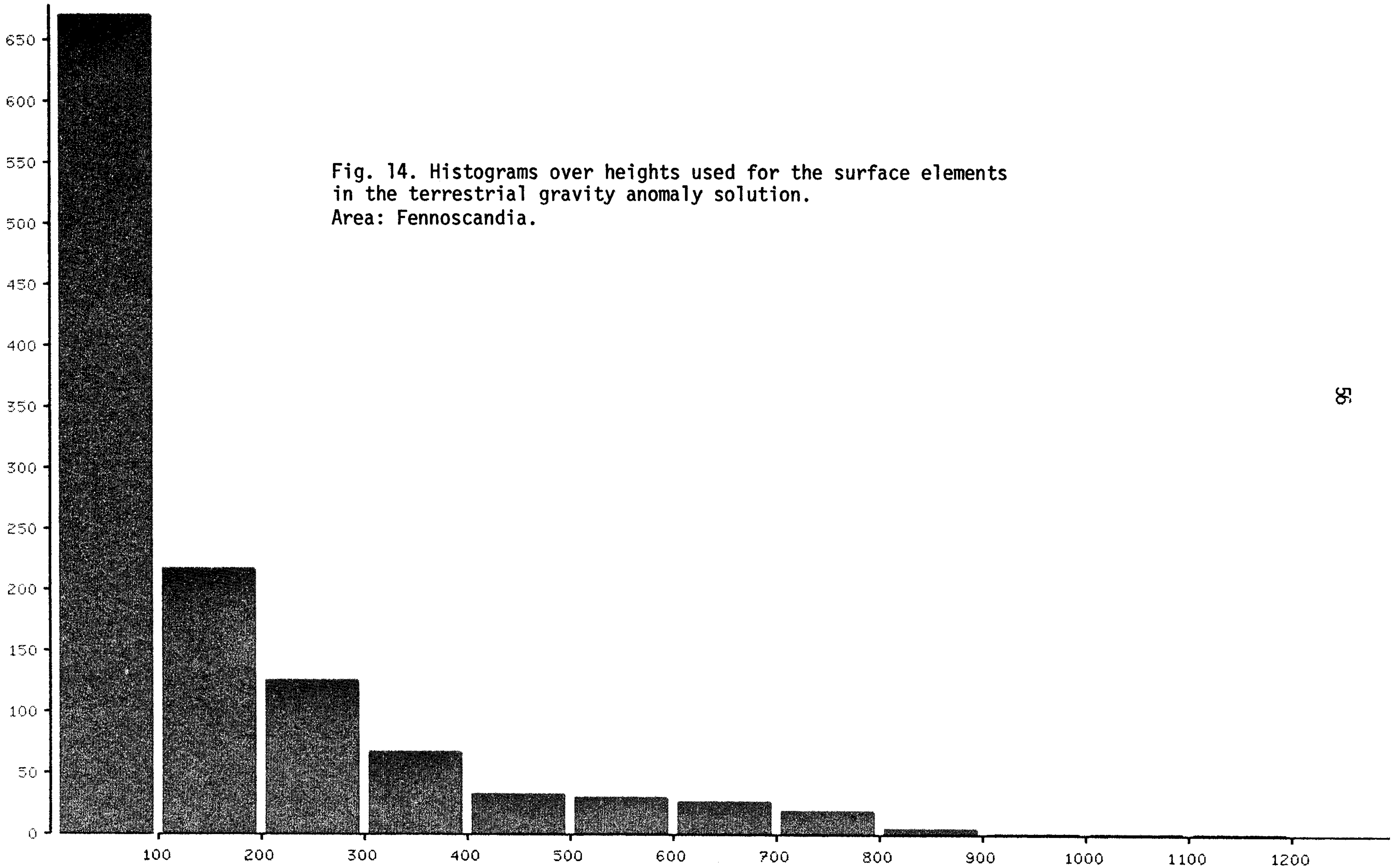


Fig. 14. Histograms over heights used for the surface elements in the terrestrial gravity anomaly solution.
Area: Fennoscandia.

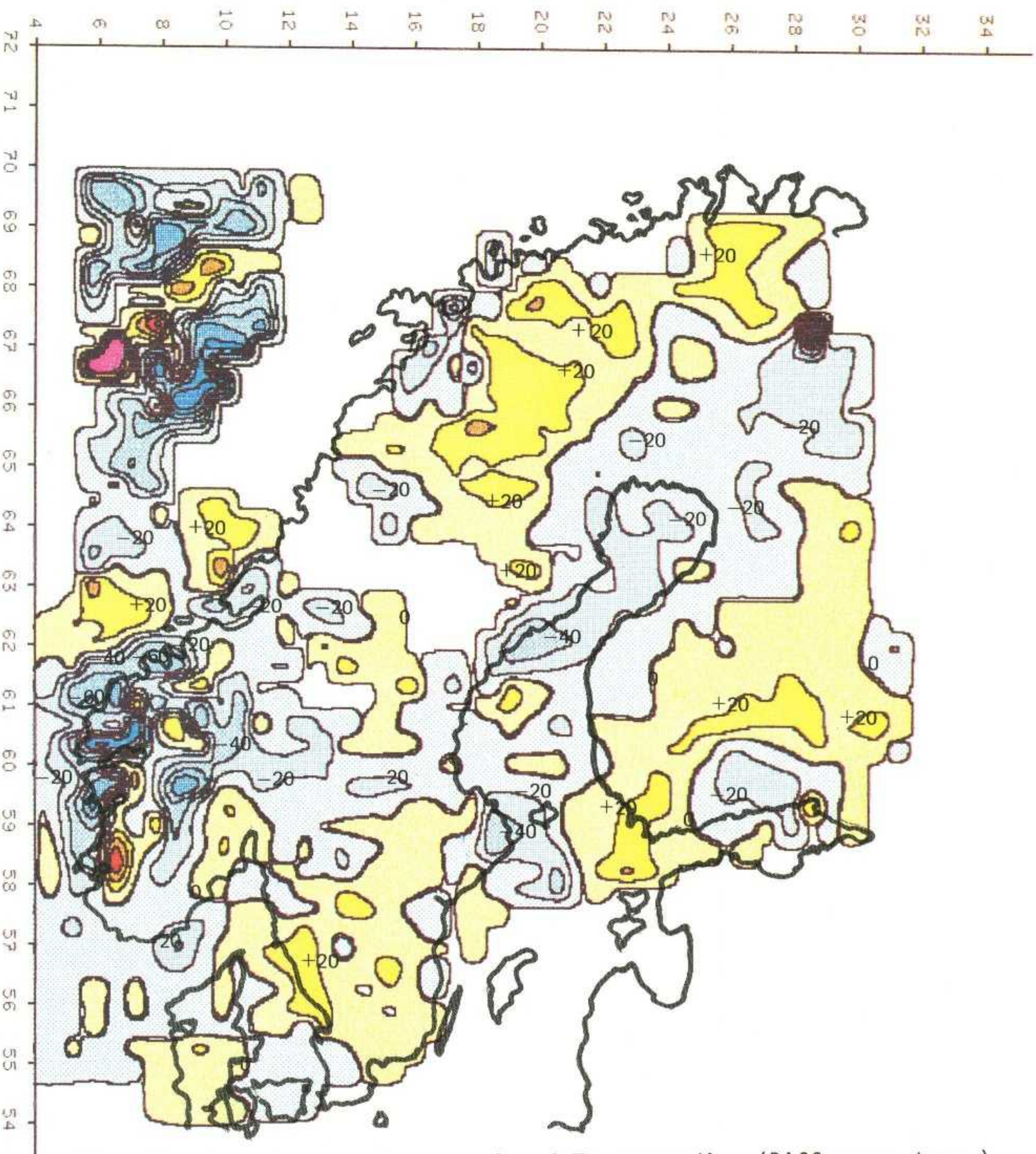
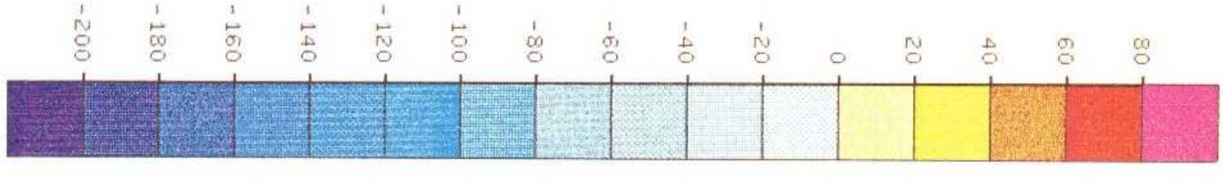


Fig. 15. Free air gravity anomaly of Fennoscandia. (Difference type.)
Anomaly defintion: Observed free air anomaly (terrestrial) - satellite free air anomaly.
Spherical harmonic degrees: > 30.
Unit: 1 milligal.



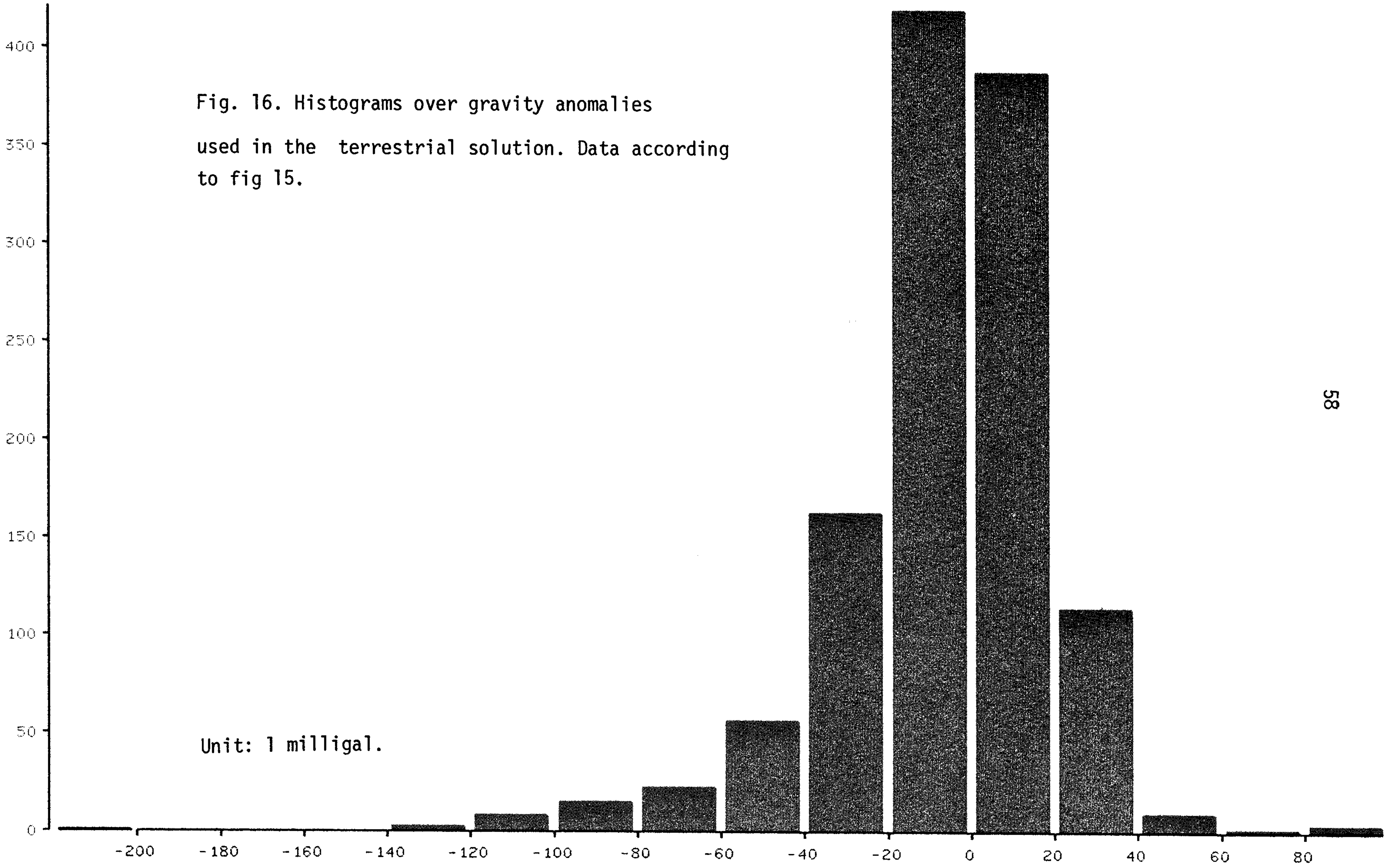


Fig. 16. Histograms over gravity anomalies used in the terrestrial solution. Data according to fig 15.

Unit: 1 milligal.

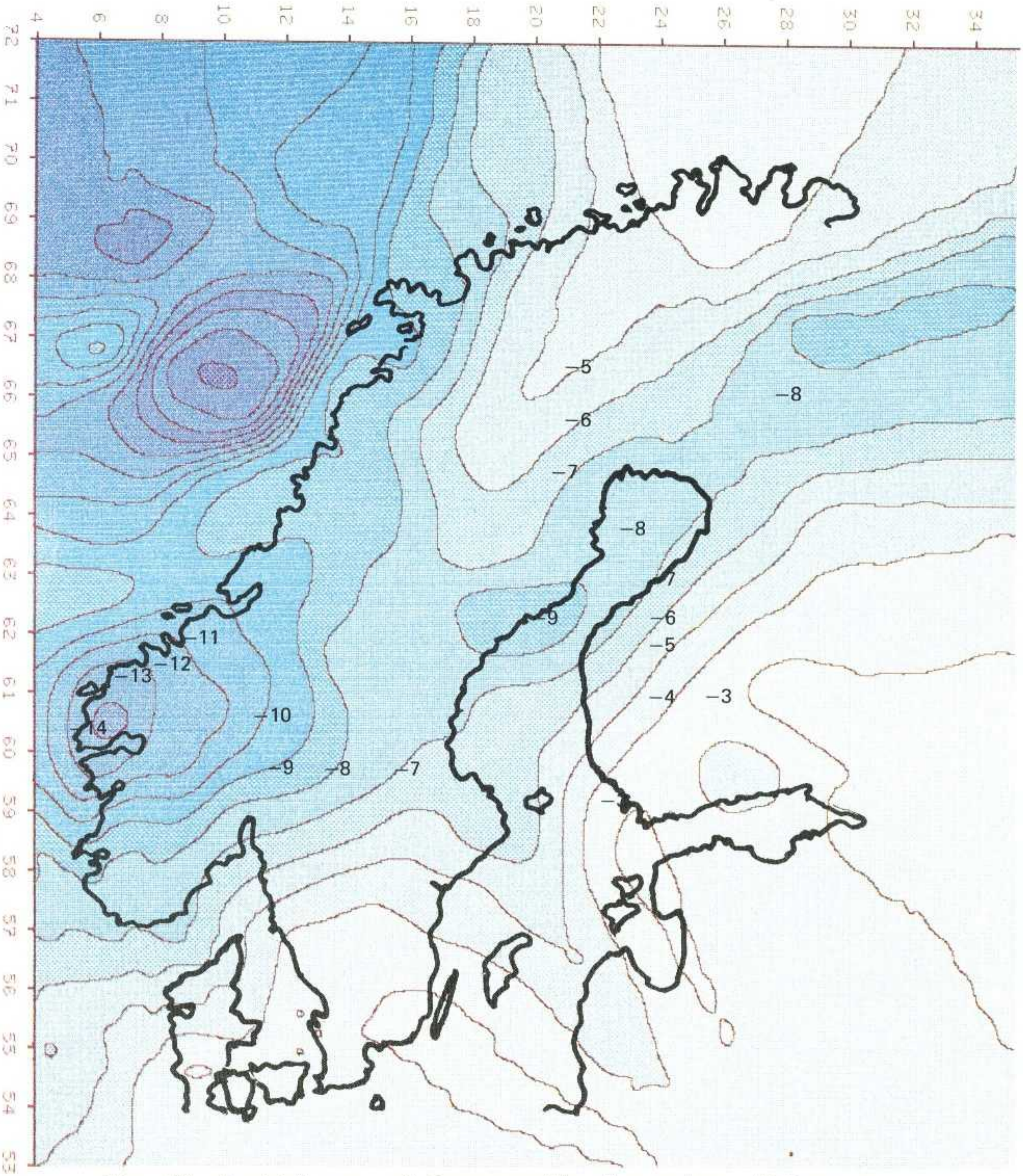
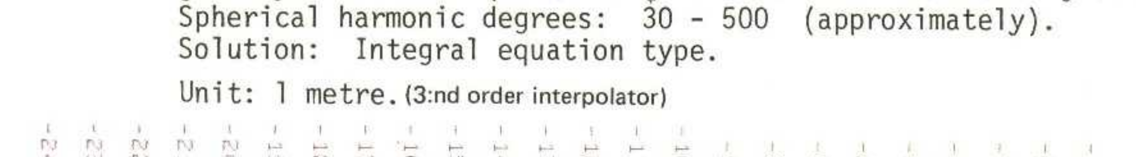


Fig. 17. Geoidal map of Fennoscandia when using difference gravity anomalies. (Geoid heights above the "satellite geoid".) Spherical harmonic degrees: 30 - 500 (approximately). Solution: Integral equation type.

Unit: 1 metre. (3:nd order interpolator)



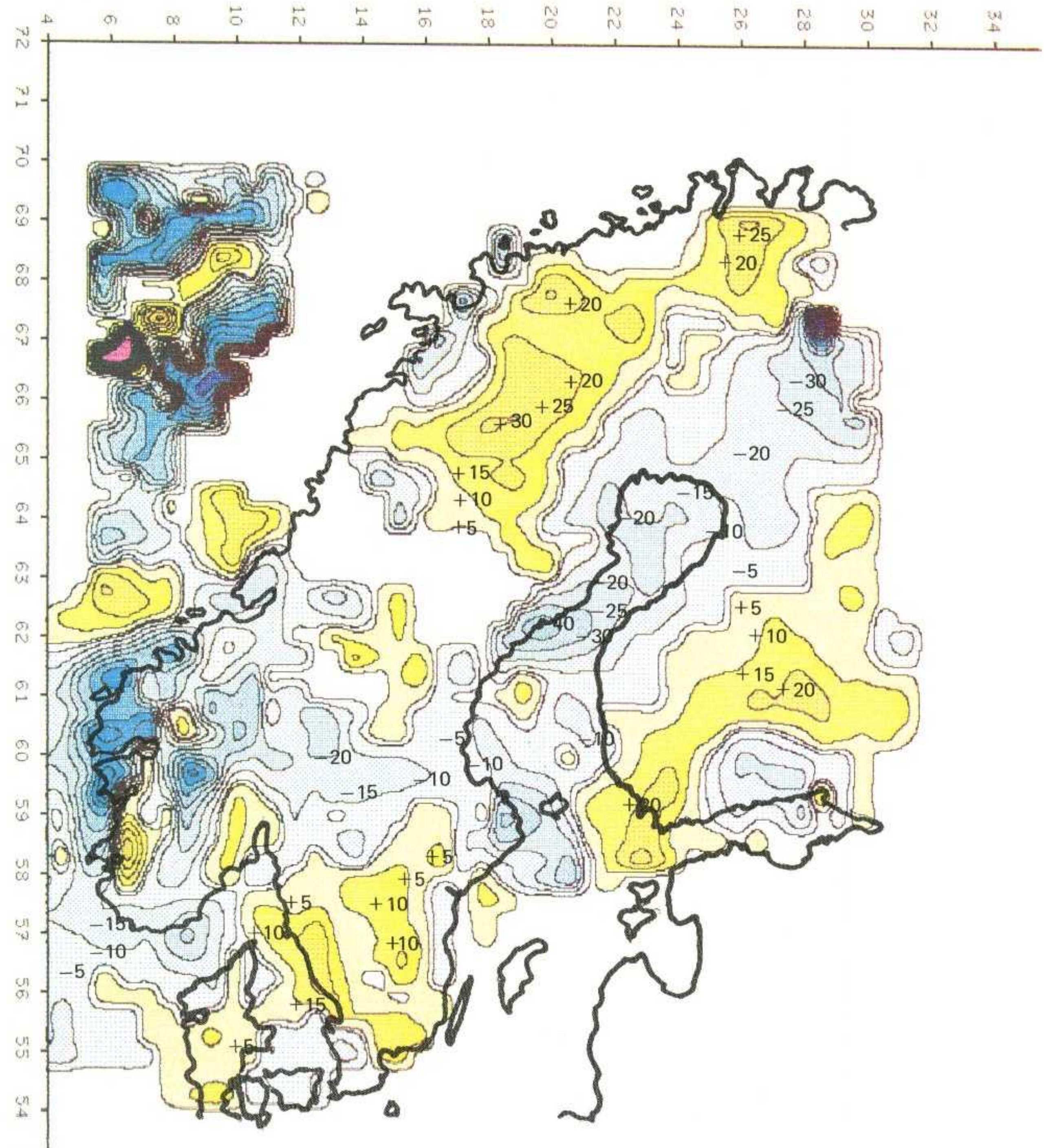
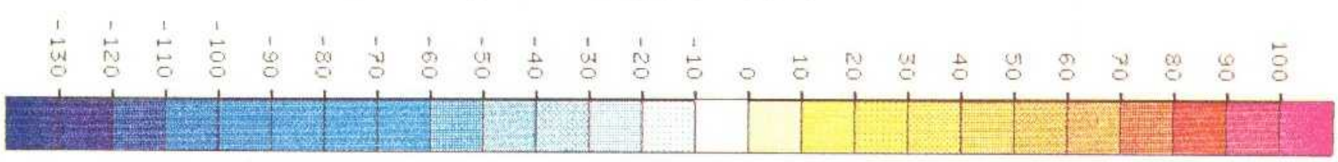


Fig. 18. Predicted free air anomalies obtained from the solution of an integral equation with 1597 unknowns.
Conclusions: There is a very good correlation with the original difference anomalies (fig 15). This map has half the equidistance of map on fig 15. Unit: 1 metre.



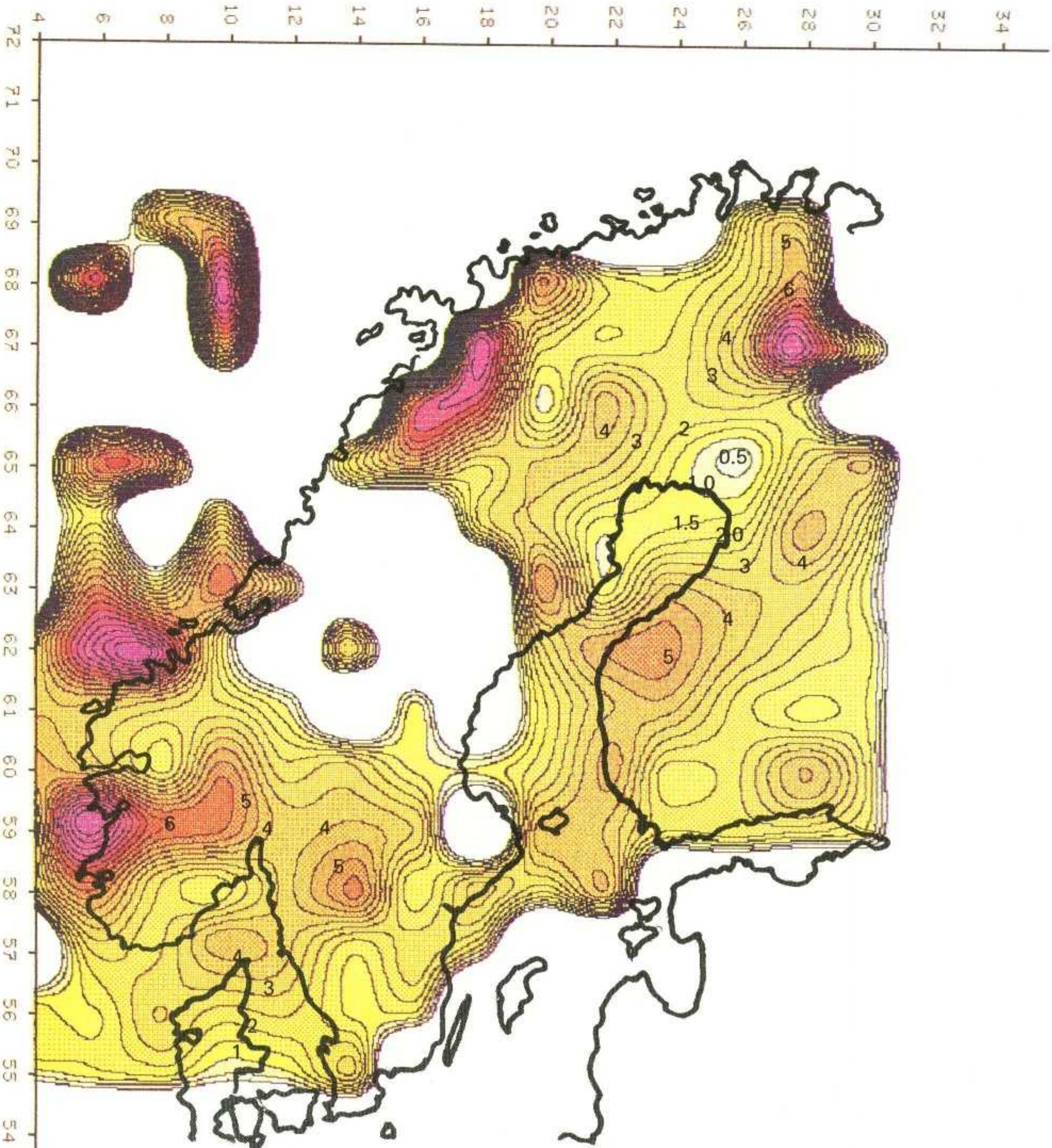
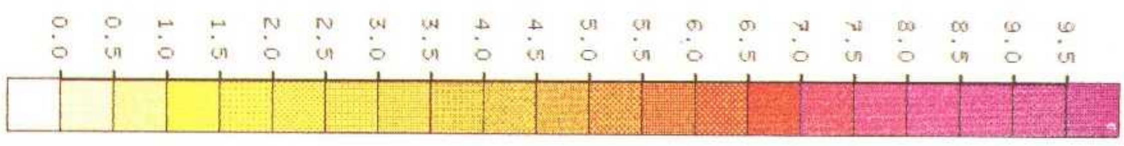


Fig. 19. Vertical deflections (magnitude) from the solution with the difference gravity anomalies in fig 15. Unit: 1".



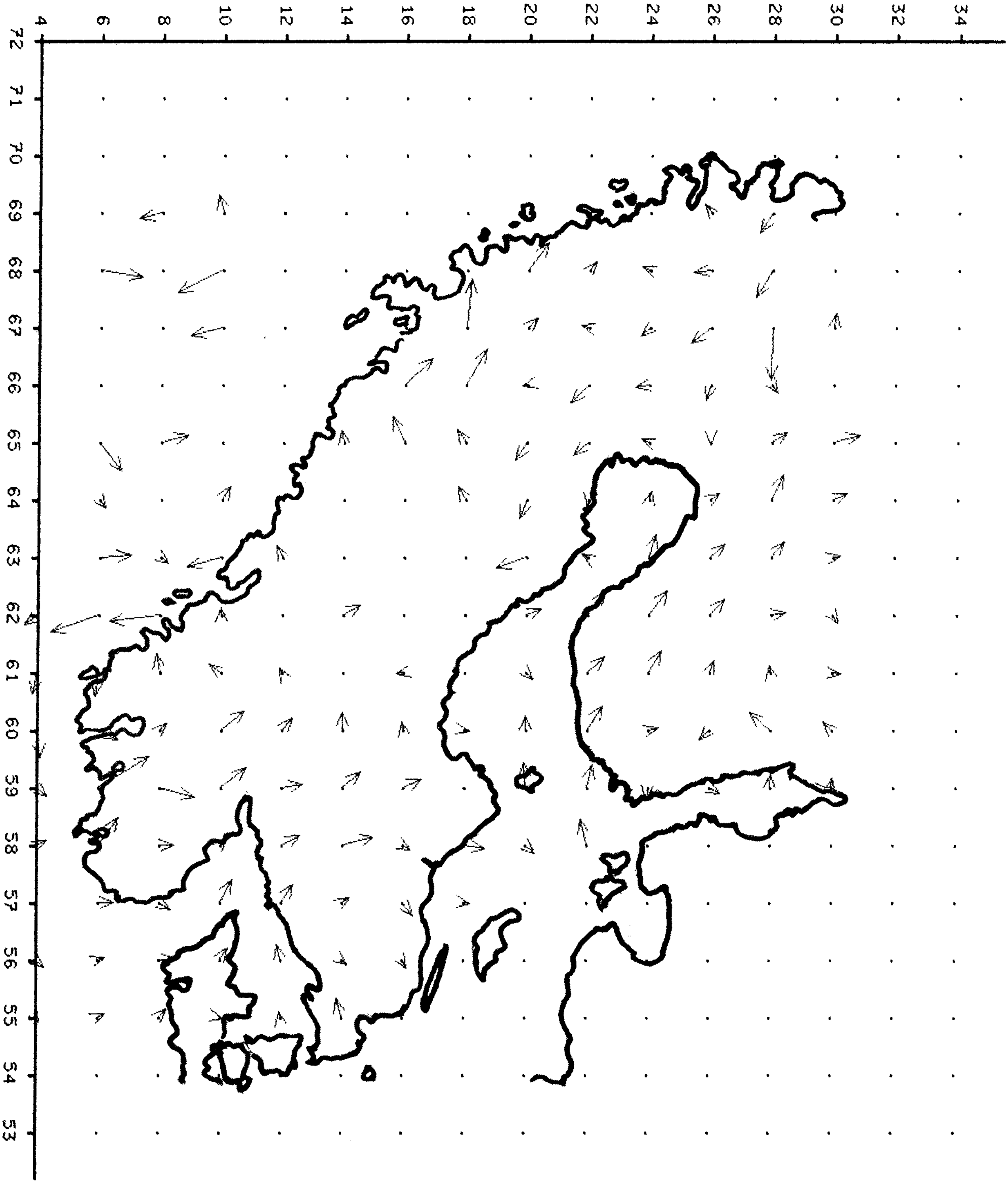


Fig. 20. Residual horizontal force field in Fennoscandia.
Mapped quantity: Direction and magnitude of the horizontal vertical deflection. Unit 1"/mm.
Data: See fig 15.
Conclusions: Some areas have excessive horizontal "tensions".
Note: Satellite data are not included in this solution. The total horizontal force field is obtained after adding the contribution from the low order solution.

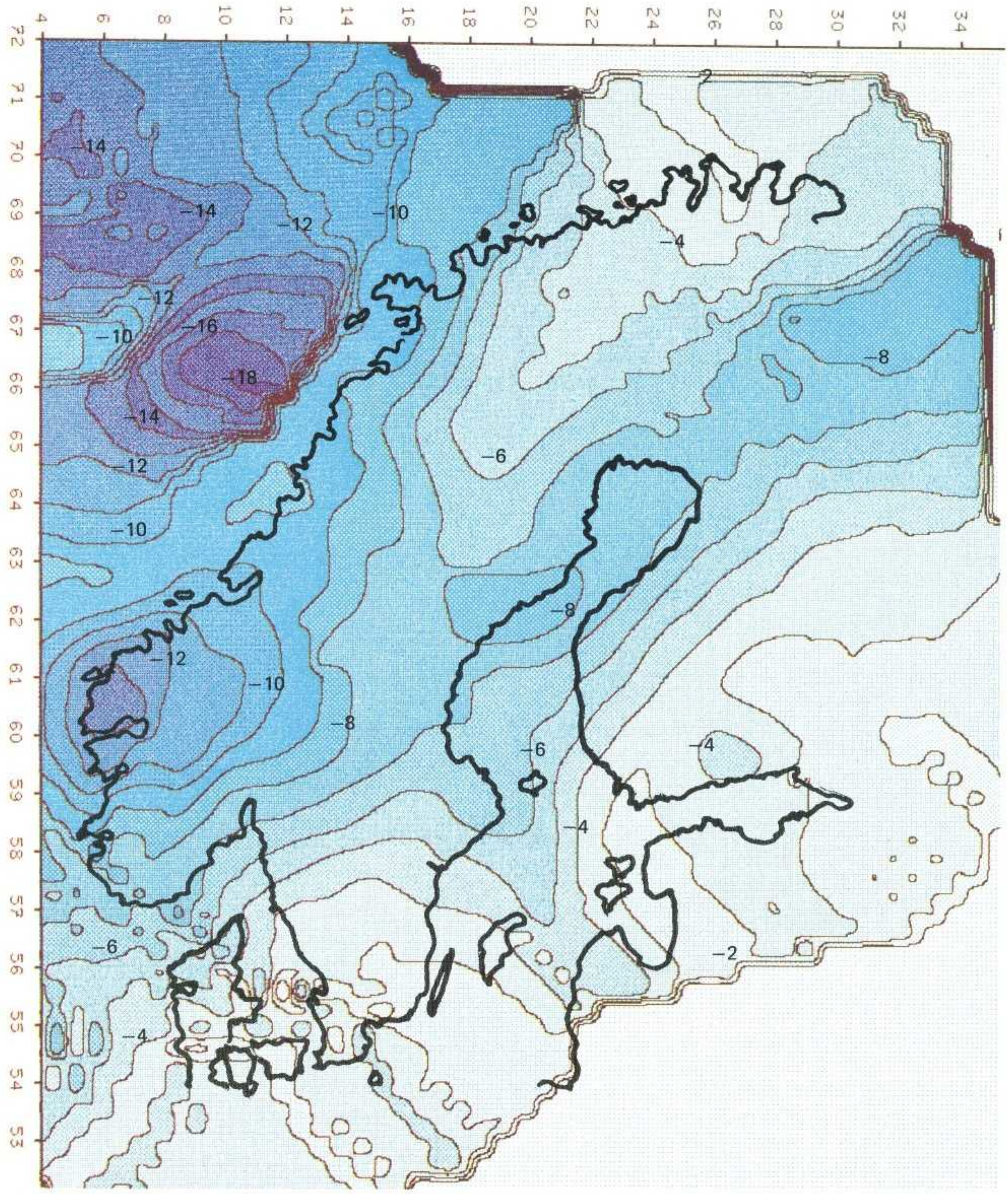
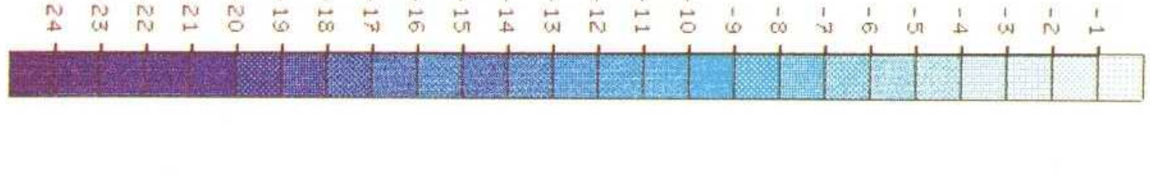


Fig. 21. The map in fig 17 with interpolation over the unsurveyed areas. (2:nd order interpolator)



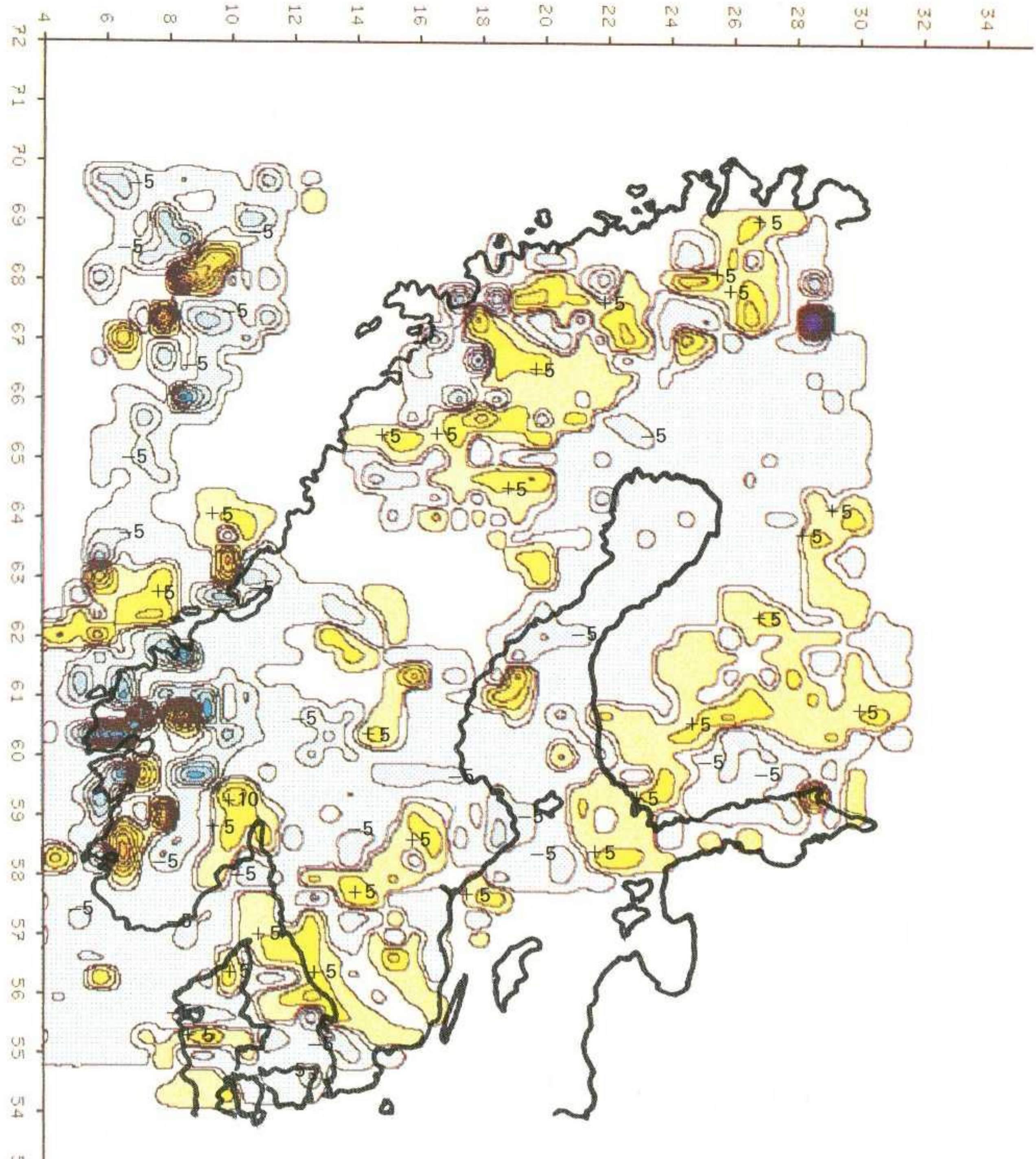
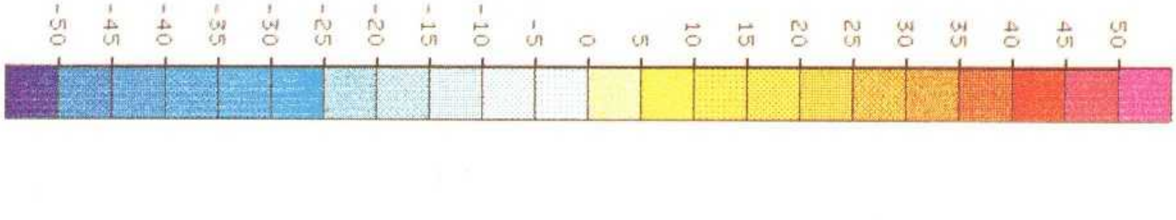


Fig. 22. Final residual gravity anomalies.
Spherical harmonic degrees: > 500 (approximate).



14. REMAINING UPLIFT FROM GRAVITY SOLUTION

An estimate of the remaining uplift can be obtained from the two gravity solutions. The total subsidence was determined to 18 m for the peak value of the Fennoscandian geoid, when only considering the isostatic contribution from the glaciated area. We can estimate a lower limit for the remaining uplift in the following way.

The earth is considered spherical and the subsidence of the geoid is caused by a circular sheet of constant height h. The density of the mantle is ρ and the geocentric angle of the circular sheet is θ.
We postulate that the geoidal subsidence is caused by a subsidence in the crust and in the mantle. The density of the mantle is considered valid for circular sheet that creates the geoidal subsidence. We obtain the disturbance potential (T) for this sheet (r_0= radius of the earth)

$$T = 4\pi G \rho h \sin(\theta/2) r_0$$

The geoidal subsidence is then (N)

$$N = T/g$$

where g is gravity.

The height of the subsidence sheet will then be

$$h = Ng/4\pi G \rho \sin(\theta/2) r_0$$

This is a lower limiting value. The sheet will probably reduce in height with increasing distance from the centre and h will increase in magnitude.

The subsidence sheet represents the remaining uplift. We obtain for $\theta = 10^0$ $\rho = 3280$ (MKS) and $N = 9+9 = 18$ m

$$h = 116 \text{ m}$$

This figure should be compared with the corresponding figure from the analysis of the uplift records in Ångermanland (see chapter 10). There we found the remaining uplift 105 m. This means that the two completely different types of solution are fully compatible.

15. CONCLUDING REMARKS.

We have analysed all availabe information in a study of the Fennoscandian crust with respect to present movements in the crust. The main object of our investigation has been to discriminate between tectonic and isostatic movements.

We have analysed all the evidence we have found in favour for the tectonic hypothesis and we found little support for this alternative. In fact, there has been practically no proof for the existence of a major tectonic movement in the Fennoscandian crust. Small tectonic movements cannot be excluded, but present observations are not sufficient for a conclusive statement.[x)]

The main interest of our study has been devoted to the analysis of the gravity field with the use of satellite observations and terrestrial gravity observations. This study has revealed that there is a very strong support for the hypothesis that the present vertical movements in the crust have an isostatic origin. We find it also justified to conclude that the movements are mainly of glacioisostatic origin.

This study has not considered the tidal movements in the crust.

The result of the analysis seems to imply that no alarming tectonic movements in Fennoscandia can be traced from the records now studied.

The problem of the optimal choice of a site for waste disposal inside the Swedish crust is very intricate and must be considered from a number of different points of view.

x) Records from Rheingraben indicate a tectonic uplift of maximum 1.7mm per year. The seismic activity in this area is stronger than in Fennoscandia and we estimate the present tectonic uplift in Fennoscandia to be below 1 mm per year!

If radioactive waste has to be disposed in the crust, then the following priorities will be given.

1. Salt mines.
2. Permafrost-areas.
3. Selected areas, which are 'stable' from geodynamical point of view.

The two first priorities are fairly equivalent with respect to the environmental problems. The waste disposal should not endanger any life for any estimable time span. However, the first alternative is excluded in Fennoscandia. The second alternative is a possible alternative for some rare locations in northern Sweden. There is a practical problem with the heat dissipation, which means that the final disposal must be postponed considerable time (several hundred years).[x)] Meanwhile, a temporary waste disposal can be arranged above sea level in dry locations.

The third alternative has the lowest priority, but is the most convenient for the user. If this alternative is chosen, then it is recommended that an advanced geodynamical study is made over a time span of at least 10 years for the selected sites. Crustal studies of the horizontal and vertical movements should be made. The seismic activity in the neighbourhood of the sites should be studied. Elastic, viscoelastic and viscous parameters should be determined with great care. The final sites should be chosen in areas where the isostatic relaxation has been more or less completed.

Finally, it should be noted that large postglacial horizontal movements in Fennoscandia have recently been recorded by Lagerbeck and Lagerlund. These findings might indicate that the crustal movements are more complex than earlier anticipated.

x) Eventually restricted to transuraniums, in several small containers.

APPENDIX 1

A REVIEW OF DISCRETE METHODS IN PHYSICAL GEODESY

The analytical background

Different methods have been used for a solution of the boundary value problem in physical geodesy. The classical solution according to Stokes anticipated that the earth was a spherical body and that all masses were located inside the sphere. The geodesists tried to develop reduction techniques which made it possible to eliminate the influence of external masses or reduce the errors caused by these masses. Various reduction techniques were in use and we can mention the methods of Pratt, Airy, Heiskanen and Rudzky. Molodensky (1948), gave the foundations for a new geodesy which had the ambition to give a strict solution of the boundary value problem for a non-spherical surface. All these methods were analytical. Hörmander (1976), made a careful study of the existance problem in the analytical case. He proved that there is a solution, at least when the topography is fairly smooth. (Hölder class $H^{2+\varepsilon}$.) This means that it is still an open question if the principal boundary value problem of physical geodesy has a meaningfull solution for a "geodetic topography".

The analytical solution of the boundary value problem is somewhat academic in geodesy because we only know gravity at discrete points on the surface of the earth (and in the space). Therefore, every practical solution has also to include a prediction technique, that defines all the missing boundary values. This prediction problem is left outside the solution in all analytical procedures.

1. THE DISCRETE APPROACH

A fully discrete approach was described in 1963 (cf. Bjerhammar 1963). The presented solution can be described in the following way.

1. Gravity anomalies are given at N discrete points on the surface of the earth (or in the space)
2. An internal reference sphere is defined. (This "geo-sphere" should have its centre in the gravity centre of the earth.)

3. It is wanted to find a gravity distribution on the sphere, that satisfies the given boundary values when the Laplace condition is valid outside the sphere.

4. The solution of the boundary value problem for the sphere is then computed for all points at the surface of the earth and in the space.

5. All missing boundary values are indirectly defined in an analytical way.

This type of solution was of course controversial in many respects. The main objections can be summarized thus:

1. Geophysical facts make it clear that the Laplace condition should not be satisfied between the surface of the earth and the internal sphere. We know, however, that there is an infinite number of mass distributions which all will satisfy the given boundary values.

2. The existence of a solution has been questioned.

Definitions:

1. Collocation is a discrete method of solving integral- and differential equations.

2. Translocation is a collocation where the solution is related to an auxilliary reference surface in such a way that the Laplace condition is satisfied outside this surface.

In the classical collocation technique, it is only requested to find a solution of an integral equation (or differential equation) with the use of discrete procedures. In our approach, we have included additional constraints.

We have called this procedure translocation indicating that the masses outside the sphere are translocated into the sphere before a solution with collocation is made. Krarup (1969) described the method as a collocation. The comments by Krarup indicate that he was originally rather critical against the method. Finally he verified with the use of a generalized Runge-theorem the existence of solution. See also Keldych and Lavrentieff (1937), who studied the existence problem in a rather general way.

It is of course not sufficient to verify that a problem has a solution. It is also important to find a useful solution. We will here give a presentation of some of the solutions of the "translocation problem".

We use the following relations between the gravity anomalies at the surface of the earth Δg and the reduced gravity anomaly Δg^* at an internal reference sphere. (It is here inticipated that $r_j \Delta g_j$ can be considered to be harmonic.)

$$\Delta g_j = \frac{r_j^2 - r_o^2}{4\pi\ r_j} \iint \frac{\Delta g^*}{r_{ji}^3}\ dS$$

Δg_j = gravity anomaly (at the surface of the earth or in the space).

Δg^* = reduced gravity anomaly at the sphere

r_j = geocentric distance to the actual point

r_o = radius of the sphere

r_{ji} = distance between the actual point (P_j) at the physical surface and the moving point (P_i) at the sphere

S = surface of integration

This type of relation was originally given by Poisson for the computation of an harmonic function outside a spherical body.

We have modified the formula and use it in an integral equation for the computation of Δg^* when Δg is known at a non-spherical surface.

1. Grid-method. (Bjerhammar 1963, 1964, 1968, 1969, Sjöberg 1975)

Surface gravity data are given in arbitrary positions (normally grid positions). Mean values of gravity anomalies Δg are computed for each surface element. The centre of the surface element represents this element with a gravity value equal to the mean value for the whole element. A second grid system is defined on the sphere. It is anticipated that gravity Δg^* is constant inside each "surface" element. Some alternatives are obvious.

2. Overdetermined system: number of Δg-values = n
number of Δg^*-values = m

The integral equation $n > m$ is represented by the linear matrix equation

$$A \Delta g = \Delta g^*$$

and we have the unique least squares solution (for full rank of A)

$$\Delta g^{*} = (A^{T}PA)^{-1}A^{T}P\Delta g = A_{OP}^{-1}\Delta g \ , \quad P^{-1} = E\{\Delta g \Delta g^{T}\}$$

3. Underdetermined system: $n < m$

Minimum norm solution

$$\Delta g^{*} = A^{T}(AA^{T})^{-1}\Delta g = A_{IO}^{-1}\,\Delta g$$

minimizes $(\Delta g^{*})^{2}$

4. Non-singular system $|A| \neq 0$.

$$A\ \Delta g^{*} = \Delta g$$

$$\Delta g^{*} = A^{-1}\Delta g$$

If the diagonal elements of the A-matrix is larger than the sum of the remaining elements in the row, then the system has a unique solution. (Note. This is not an iff-condition.)

5. Fully degenerative system $|A^{T}A| \neq 0, \quad |AA^{T}| \neq 0$

$$A\Delta g^{*} = \Delta g$$

$$\Delta g^{*} = A_{IP}^{-1}\,\Delta g, \qquad A_{IP}^{-1} = A_{IO}^{-1}\,AA_{OP}^{-1}$$

Unique least squares solution with

$$(\Delta g^{*})^{2} = \min \qquad (A\Delta g^{*} - \Delta g)^{T}\,(A\Delta g^{*} - \Delta g) = \min$$

This solution is unique for any A when $|P| \neq 0$

Special cases.

6. Hilbert space solution (Krarup 1969)

If $m \to \infty$ (and well-behaved surface elements) then the solution 1.2 will have the Hilbert space solution with reproducing kernel as a limiting case. See Sjöberg (1975).

$$\iint (\Delta g^{\bullet})^2 dS = \min \qquad \text{or alternatively}$$

$$||\Delta g^{\bullet}|| = \min \qquad (L_2 \text{ - norm})$$

7. Wiener-Hopf solution (stochastic process) (Moritz 1970)

If $m \rightarrow \infty$ then we have the Wiener-Hopf solution when minimizing $||\Delta g^{\bullet}||$.

It has been shown by Parzen (1961) that there is a dualism between the Wiener-Hopf method and the Hilbert space solution. However, the Wiener-Hopf approach anticipates some additional stochastic properties.

The observations belong to a weakly stationary stochastic process, wich here means that the expectation

$$E\{\Delta g_t \; \Delta g_{t+\tau}\} = c(\tau)$$

The covariance function must be invariant with respect to the difference in time (= distance)

The stochastic process has to be ergodic, which means that it should be possible to estimate the covariance function from the observations. It has been proved that our process is not ergodic (Lauritzen 1973).

When the process is ergodic, then <u>optimal</u> solution can be determined.

We have no direct probability justification for any selected covariance function in our application. The method is of course still justified as a purely deterministic approach.

Moritz (1970) used the following covariance function (minimizing the L_2-norm of Δg)

$$E\{\Delta g_i \; \Delta g_j\} = \sum_{n=0}^{\infty} \sigma_n^2 \Sigma^2 (2n-1) \; t^{n+2} \; P_n(\cos \omega_{ij})$$

σ_n^2 = degree variance

t = $r_0^2/r_j r_i$

$P_n(\cos \omega_{ij})$ = Legendre polynomial

There seems to be no practical method for a determination of the degree variances. Some newer investigations accept the method as a deterministic approach.

Wiener and Hopf have given the solution of the optimal linear prediction ($\hat{x}$) for a weakly stationary process (h).

The solution is for the discrete case

$$\hat{x} = E\{xh^T\}\left[E\{hh^T\}\right]^{-1}h$$

h = vector of the discrete observations, with $E\{h\} = 0$
$E\{hh^T\}$ = auto-covariance matrix
$E\{xh^T\}$ = cross-covariance matrix

Covariance stationarity is postulated which implies that the covariance function $c(\tau)$ is independent of absolute time (see above)

$$E\{h_{t_1} h_{t_1+\tau}\} = c(\tau)$$

h_{t_1} = the stochastic process at the time t_1

$h_{t_1+\tau}$ = the stochastic process at the time $t_1+\tau$
etc.

This process was originally developed for time series but applications to other series have been frequently used. (The time parameter has been replaced by the distance parameter in geodetic applications.)

This solution is only valid if the stochastic process is "ergodic" which means that the observations can be used for an estimation of the covariance function.

It is well-known from a number of practical applications that a strict computation of the covariance function is seldom possible because the processes are not ergodic and therefore, not directly useful for a determinations of the covariance function.

2. REFLEXIVE PREDICTION

Problem: In a stochastic process a set of observations $h_1, h_2 \dots h_n$ are given. It is requested to find a set of unknowns $x_1, x_2 \dots x_m$ having known "time" parameter $t_1, t_2 \dots t_m$ and a given covariance function of the unknowns, where the set of unknowns have the given observations as optimal predictions for the prescribed covariance function.

Theorem: The optimal linear predictor is for $m = n$

$$h = E\{hx^T\} \; (E\{xx^T\})^{-1} x \qquad (1)$$

Proof. The optimal linear predictor has to minimize the variance of the prediction. We have

$$h + \varepsilon = (A + \Delta A)x \qquad (2)$$

where A = the linear predictor $(n \times 1)$
ε = observation error (1×1)

$$\varepsilon\varepsilon^T = A\, xx^T A^T + \Delta A\, xx^T \Delta A^T + 2A\, xx^T \Delta A^T + \\ + hh^T - 2hx^T A^T - 2hx^T \Delta A^T \qquad (3)$$

$$E\{\varepsilon\varepsilon^T\} = A\, E\{xx^T\}\, A^T + \Delta A\, E\{xx^T\}\, \Delta A^T + \\ + 2A\, E\{xx^T\}\, \Delta A^T + E\{hh^T\} - 2E\{hx^T\}A^T - 2E\{hx^T\}\Delta A^T$$

This expression has a minimum if the non-symmetrical terms vanish, or

$$(EA\{xx^T\} - E\{hx^T\}) = 0 \quad \text{and}$$

$$A = E\{hx^T\} \cdot (E\{xx^T\})^{-1} \qquad (4)$$

Then

$$E\{hx^T\}A^T = AE\{xx^T\}A^T = \text{symmetrical}$$

Using this A we find the variance for $A + \Delta A$

$$E\{\varepsilon\varepsilon^T\} = E\{hh^T\} - A(E\{xx^T\})^{-1}A^T + \Delta A(E\{xx^T\})^{-1} \Delta A^T \qquad (5)$$

This variance has a minimum iff $\Delta A = 0$ for full rank of $E\{xx^T\}$.

The optimal prediction (minimum variance solution) is

$$\underset{n1}{\hat{h}} = \underset{nn}{E\{hx^T\}} \; \underset{nn}{(E\{xx^T\})^{-1}} \underset{n1}{x} \tag{6}$$

We introduce the parameter y

$$y = (E\{xx^T\})^{-1} x = Q^{-1}x \;, \qquad Q = E\{xx^T\} \tag{7}$$

$$x = Qy$$

and obtain the unique solution

$$\hat{h} = E\{hx^T\}\; y = Ry \quad \text{where} \quad R = E\{hx^T\} \tag{8}$$

Any new prediction z is

$$z = E\{zx^T\}\; E\{xx^T\}^{-1}x = E\{zx^T\}y \tag{9}$$

F i l t e r i n g

If $m < n$ then we have the least squares solution

$$y = (R^TR)^{-1} R^Th = R_{0I}^{-1} h$$

which minimizes

$$(h - Ry)^T (h - Ry)$$

This type of filtering gives dramatic reduction in the computitional work compared with the Wiener-Hopf approach.

Note. The Wiener-Hopf approach is obtained as a special case for $m = n$ and carrier points in the given observation points. There is a corresponding loss of stability in the solution because the condition number is approximately squared. The computations technique used with generalized inverses will sometimes be needed.

3. DIRAC APPROACH

We choose the surface elements in the grid method in such a way that $\Delta g^{\bullet} = 0$ all over on the geo-sphere, with the exemption of a number of surface elements which all have infinitesimal size. Using this technique, we can exclude all integrations and obtain a fully discrete approach

$$\lim_{\Delta s_n \to 0} \frac{r_j^2 - r_o^2}{4\pi r_j} \iint \frac{\Delta g_n^{\bullet}}{r_{ji}^3} dS = \frac{c(r_j^2 - r_o)r_o^2}{r_j r_{ji}^3} \Delta g_n^{\bullet}$$

$$c = \text{constant}$$

Here $\Delta g_n^{\bullet}$ represents a Dirac impulse and we obtain a system of linear equations which replacesthe original integral equations in a strict way. We note that this type of solution is fully analytical outside the geo-sphere. (It is fully equivalent with the Hilbert space solution in the two-dimensional case when using equal spacing.)

The equivalence with the reflexion prediction with carrier points on the geo-sphere is trivial.

4. MEAN VALUE APPROACH

Using this technique we choose the surface elements of our grid technique in a rather special way. The size of the surface element is not defined and we only postulate that gravity is constant inside the surface element. Then we can use the mean value theorem of integral calculus and select a point which has to represent the element. The mean value theorem states that there is a point (inside each finite surface element), that can represent the whole surface element in a fully discrete way. This means that we don't need to specify the size of the surface element. We simply let it be undefined and make an indirect determination by the relation

$$\frac{r_j^2 - r_o^2}{4\pi r_j} \iint_{\Delta S_n} \frac{\Delta g^{\bullet}}{r_{ji}^3} dS = \frac{(r_j^2 - r_o^2)r_o^2}{r_j r_{ji}^3} \Delta g_n$$

We note that the surface elementscan here be partly overlapping.

5. LEAST SQUARES COLLOCATION

Krarup (1969) introduced a new technique for combined least squares adjustment and prediction according to Wiener-Hopf.

For the least squares adjustment we have the classical approach.

Stochastic model

$$AX = E\{L\}$$

where A = (nxm) matrix of known quantities
X = (mxl) vector of unknown parameters
L = (nxl) vector of stochastic variables

The Gauss-Markow type of least squares solution is

$$\hat{X} = (A^T PA)^{-1} A^T PL$$
$$V = A\hat{X} - L$$

minimizing

$$V^T PV$$

where

$$P = [\mathrm{cov}\{L\}]^{-1} = E\{\varepsilon\varepsilon^T\}$$

$$\varepsilon = L - E\{L\}$$

The Wiener-Hopf approach for the discrete case can be describe in the following way.

If the vector V represents a set of stochastic variables in a weakly stationary stochastic process, then the optimal prediction $\hat{Y}$ is according to Wiener-Hopf

$$\hat{Y} = E\{YV^T\}|E\{VV^T\}|^{-1} V$$

where

$E\{v_{t+\tau}\ v_t\} = c(\tau)$ = covariance function
$v_{t+\tau}, v_t$ = elements of V
t, τ = "time" parameters

The Gauss-Markow least squares technique and the Wiener-Hopf approach can be directly combined in the following way:

1:o A classical least squares solution (best linear unbiased estimator) is first determined from the given observations with the use of the known covariance matrix.

2:o The residuals are computed and the best covariance function for these residuals is determined.

3:o The Wiener-Hopf predictions are computed.

We anticipate that the observations include a random noise ε and a signal s. If the noise and the signal are uncorrelated, then we have

$$L = E\{L\} + \varepsilon + s$$

and

$$\text{cov}\ \{L\} = E\{\varepsilon\varepsilon^T\} + E\{ss^T\} = R + S$$

$$R = E\{\varepsilon\varepsilon^T\} \qquad S = E\{ss^T\}$$

This means that the Gauss-Markow/Wiener-Hopf solution is

$$\hat{X} = |A^T(R + S)^{-1}A|^{-1}\ A^T(R + S)\ L$$

$$\hat{Y} = q_{YV}\ (R + S)^{-1}V \qquad q_{YV} = E\{YV^T\}$$

In the practical application, we have to find a suitable technique for a determination of the covariance matrix of the observations. If this covariance matrix is correctly determined, then is identical with R + S. There are some difficulties in the determination of $(R + S)^{-1} = P$ and this quantity is often considered to be an a priori quantity without separation between R and S.

Krarup gave his solution in a more general way with the use of integral operators.

The main difficulties in the practical application of this technique are found in the determination of the covariance function of the residuals (V). If we have an ergodic process, then it should normally be possible to estimate the covariance function from the observations. This is hardly possible here because the covariance function has to be determined before the residuals have been defined.

The difference between the two different approaches is somewhat formal.

Wiener-Hopf

The covariance function is estimated from the observations and the optimal predictions are then computed.

Reflexive prediction

The covariance function is given "a priori" and we compute a set of "observations" which give known observations as optimal predictions for the actual covariance function.

A critical reader can ask why we don't accept the "a priori" covariance function for the Wiener-Hopf approach. The answer is of course that, the optimal predictions are undefined in the non-ergodic case. (Improperly posed problem.)

The philosophy of the reflexive prediction is slightly different then the Wiener-Hopf approach. It is obvious that the Wiener-Hopf approach is imroperly posed problem for the boundary value problem of physical geodesy. When using the reflexive prediction approach, then we have modified the problem in sach a way that we have a properly posed problem.

1:o The covariance function is considered known

2:o It is requested to find a set of fictitious observation which have the given observations as optimal predictions for the given covariance function

This problem is now properly posed.

6. KRONECKER APPROACH (Rauhala 1976)

Rauhala (1976) wrote: "I am really serious when contrasting the computation time of 10^6 years of the conventional method and 10 minutes of the array case."

Rauhala used the Kronecker multiplication laws on the matrix equation

$$\underset{NN}{A} \; \underset{N1}{X} = \underset{N1}{\Delta g}$$

and obtained

$$\underset{n_1 n_2}{A_1} \; \underset{n_2 n_2}{X} \; \underset{n_2 n_2}{A_2} = \underset{n_1 n_2}{\Delta g} \qquad n_1 + n_2 = N$$

With the simple solution

$$X = A_1^{-1} \, \Delta g \, A_2^{-1}$$

Rauhala used the covariance function

$$f = \left(\frac{a}{a + (x - x_i)^2 - (y - y_i)^2}\right)^{\gamma} \qquad \gamma = 3/2$$

Then he made the factorization

$$f = \left(\frac{a_1}{a_1 + (y - y_{i_1})^2}\right)^{\gamma_1} \left(\frac{a_2}{a_2 + (x - x_{i_2})^2}\right)^{\gamma_2}$$

However, the problem of linear factorization of the covariance function for gravity has no solution (improperly posed) for the correct covariance function. The main justification for the method is the simplicity. A practical limitation is of course that the observations should be given in a grid with constant spacing and that no missing points are allowed

Rauhala found, in an numerical example, that there was no statistically significant difference between the strict solution and his own approach. It is not obvious that this type of solution is more accurate than a classical solution with integration methods. Further investigations should be of great value.

If we accept his criterium for the definition of a competitive method, then the natural compatitive estimator is

$$\Delta g_j = \Sigma(\Delta g_i/r_{ji}^3)/)1/r_{ji}^3) \qquad \text{(Bjerhammar 1968)}$$

or the slightly modified predictor

$$\Delta g_j = \Sigma e^{-cr_{ji}} \Delta g_i \qquad \text{(Jenkin type)}$$

These predictors are very much faster and simpler and the predictions are normally excellent. The prediction of gravity has been found slightly better than the Wiener-Hopf type (and reflexive) for all test examples we have tried! See chapter 9.

7. ITERATIVE SOLUTIONS

Iterative methods give extreme computational gain for large systems. There is a difference between the Wiener-Hopf approach and the reflexive prediction which corresponds approximately to a gain in condition number with 50%, when using the same radius of the internal sphere.

This means that the reflexive method is somewhat faster in an iterative approach.

The advantage of the Wiener-Hopf method is found in the symmetry of the matrix equation. It is therefore favourable for applications where non-iterative methods will be used.

The reflexive approach gives considerable reduction of the computational work when filtering is used. However, practical studies have shown that the least squares solutions give rather pure condition numbers and the solution will often require use of generalized inverses.

8. UNIFORM CONVERGENCE OF THE DISCRETE APPROACH

There is full equivalence between the reflexive prediction (or Dirac approach) and the Wiener-Hopf approach when using reductions between two concentric circles. The heights $(r_j - r_o)$ should be approximately two times larger for the reflexive prediction. Hörmander proved that there is a uniform convergence to the true value when there is a constant spacing between the observations and the number of observations goes to infinity. No similar prove has been presented for the spherical case. Thus, it is not quite obvious that the method is properly converging for the geodetic application. Krarup (1969) has a prove for the convergence in the three-dimensional case, but this prove refers to a different type of problem.

However, if we use the original grid approach with constant gravity inside each surface element, then the following discussion is valid.

We consider reduction between two concentric spheres and use a grid system with "constant spacing". There are N observed Δg-values at the external surface and N surface elements with constant gravity $\Delta g^{\bullet}$ at the internal sphere.

Now we obtain the following system of linear equations

$$\Delta g_j = \frac{r_j^2 - r_o^2}{4\pi\, r_j} \sum_{i=1}^{n} \iint_{\Delta S_i} \frac{\Delta g_i^{\bullet}}{r_{ji}^3}\, dS_i \qquad j = 1,2,3...n$$

The solution of this system is convergent if the diagonal term is larger than the sum of all remaining elements in the actual equation. If $n \to \infty$, then the limiting value is the Rieman integral and the solution is uniformly converging to the true value.

9. CONVERGENCE OF THE DISCRETE APPROACH

We let K be a compact set in R^n and $\Omega = \{x_j |x| < r_o\}$ a ball in the interior K^o of K. Furthermore, u is a continuous function in the complement of K^o (CK^o) which is harmonic in CK and $\{x\nu\}^N$ are points in CK^o such that $u(x\nu) = a$ is given. The prediction of u elesewhere is given by

$$\hat{u}(x) = \int P(x,y) \quad \phi(y)\, dS(y) \tag{1}$$

where P is the Poisson kernel of $C\Omega$.

In this L_2-norm solution we minimize

$$\int_{\delta\Omega} |\phi(y)|^2\, dS(y) \tag{2}$$

$$\hat{u}(x\nu) = a_\nu, \qquad \nu = 1, \ldots, N$$

Hörmander studied the case when $n = 2$, $K = \{x_i \; |x| < r$ and $x_1, x_2 \ldots, x_N$ are equally spaced on δK, $x_N = (r, 0)$. Furthermore, he used the simple relation

$$u(r \cos \theta, r \sin \theta) = e^{im\theta} \tag{3}$$

Then he found the prediction $\hat{u}(\theta)$

$$\hat{u}(\theta) = (\sum_{j\equiv m (\mathrm{mod}\ N)} \gamma^{|j|} e^{ij})/(\sum_{j\equiv m (\mathrm{mod}\ N)} \gamma^{|j|} \tag{4}$$

(j-m can devided by N)

$$\gamma = r_o^2/r^2$$

and

$$\hat{u}(\theta) \to u(x) \quad \text{as} \quad N \to \infty$$

for $|x| < r$ with u(x) any function with absolutely convergent Fourier series and r_o-constant.

These results look promising. However, we have to note that this approach is rather unrealistic because a numerical solution of this type gives quite unsatisfactory "condition numbers" (relation between the largest and the smallest eigen-values.)

In order to obtain a stable solution which can be handled by an ordinary computer we include the condition that the distance between the given points has a fixed ratio to the depth of the internal sphere.

We now consider equation (4) for a fixed m when $N \to \infty$. Terms in the nominator with $j \neq m$ represent high oscillations with a sum $\leq$x. This contribution goes weakly to zero. We obtain the weak limiting value

$$\hat{u}(x) = \lim_{N\to\infty} \gamma^{|m|} e^{im\theta} / \sum_{j=m \pmod N} \gamma^{|j|} \qquad (5)$$

Here $\gamma = r_N^2/r^2$

$r_N = r - a/N$

Thus

$\gamma^{|m|} \to 1$

$\gamma^N \to e^{-2a/r}$

$2\pi r/N$ = distance between given points

a/N = depth to the internal sphere

and the limiting value is

$$\hat{u}(x) \to e^{im\theta} / \sum_{\infty}^{\infty} e^{-2a|k|/r} = e^{im\theta}(1-e^{-2a/r})/(1+e^{-2a/r})$$

x) After division by the denominator.

or

$$\hat{u}(x) \to e^{im\theta} \tan hyp\ (a/r) \qquad (6)$$

We obtain

$$u(x) = u(\hat{x})\ \tan hyp\ (a/r)$$

or

$$\hat{u}(x)\ \cot an\ hyp\ (a/r) = u(x)$$

This mean that we have no longer any uniform convergence!

However, there is a weak limiting value which might be sufficient for a number of applications.

We find the following limits for the prediction error ε

$$\varepsilon_1 \geq (1 - \tan hyp\ a/r) \qquad (7)$$

Some numerical results are given below

	ε_1	0,037	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-12}
$2\pi r/a$		3,14	1,65	1,27	1,03	0,875	0,435

If $m \geq N$, then the prediction error is in the order of 1 and the prediction is no longer meaningful.

We now also discuss the situation for a prediction, when using a fixed r_0. If $m < N$ (and $m > 0$) and finite, then we have

$$\sum_{j \equiv m\ (\mathrm{mod}\ N)} \gamma^{|j|} = \gamma^m + \sum_{k>0} \gamma^{m+kN} + \sum_{k>0} \gamma^{kN-m} =$$

$$= \gamma^m + (\gamma^m + \gamma^{-m})\ \gamma^N/(1 - \gamma^N) =$$

$$= (\gamma^m + \gamma^{N-m})/(1 - \gamma^N) \qquad (8)$$

According to (8) we obtain the predictions

$$\hat{u}(x) \to \gamma^m (1 - \gamma^N)/(\gamma^m + \gamma^{N-m}) e^{im\theta} \qquad (9)$$

The error ε_2 of this expression is

$$\varepsilon_2 = 1 - \gamma^m (1 - \gamma^N)/(\gamma^m + \gamma^{N-m}) =$$

$$= (\gamma^{m+N} + \gamma^{N-m})/(\gamma^m + \gamma^{N-m}) \qquad (10)$$

An interpolated value has at least this maximal error and at most twice this error. (The sum of the coefficients is equal to 1.) For $N/2 < m < N$ we obtain from this formula

$$\varepsilon > \gamma^{N-m}/2\gamma^{N-m} = 1/2 \qquad (11)$$

and the prediction is useless.

For $\quad 0 < m < N/2$

$$\boxed{\varepsilon \leq (\gamma^{m+N} + \gamma^{N-m})/ \gamma^m < 2\gamma^{N-2m}} \qquad (12)$$

or

$$\varepsilon_3 < 2(r_o/r)^{2N-4m} \qquad (13)$$

We make a comparision with the classical linear interpolation between the points $2\pi\nu/N$. The spacing is now $2\pi/N$ and the maximal prediction error is

$$\varepsilon_4 = \pi^2 N^{-2} m^2/2 \approx 4{,}93 \ (m/N)^2 \qquad (14)$$

Examples

$N = 100$ $m = 10$ $(r/r_0)^{-1} = 0,5$

Linear interpolation:

$\varepsilon_4 < 4.93 \cdot 0.1^2 = 0.0493$

Predictions with fixed radius r_0

$\varepsilon_3 < 2 \cdot 0.5^{160} = 2.74 \cdot 10^{-48}$ $(r/r_0)^{-1} = 0.5$ $(2\pi r/a) = 0.209$

$\varepsilon_3 < 2 \cdot 0.9^{160} = 0.55 \cdot 10^{-8}$ $(r/r_0)^{-1} = 0.9$ $(2\pi r/a) = 0.628$

$\varepsilon_3 < 2 \cdot 0.99^{160} = 0.40$ $(r/r_0)^{-1} = 0.99$ $(2\pi r/a) = 6.28$

Note. The straight forward linear prediction (and the simple deterministic prediction Bjerhammar 1968)is 12 times more accurate then the Wiener-Hopf predictor using a depth of 63.7 km and a grid distance of 400 km !

A safe limit for the Wiener-Hopf approach is here

$$(r/r_0)^{-1} = 0.937 \quad \text{with} \quad \varepsilon_3 = 3 \cdot 10^{-5}$$

This ratio corresponds to the case

depth = grid distance

Summary. The discrete solution of the boundary value problems in physical geodesy with the use of the original grid method, Hilbert space solutions with reproducing kernels, Wiener-Hopf method, Dirac approach and reflexive predictions are discussed. The methods are fully equivalent for most practical applications. There is a difference with respect to the condition numbers which sometimes will be important.

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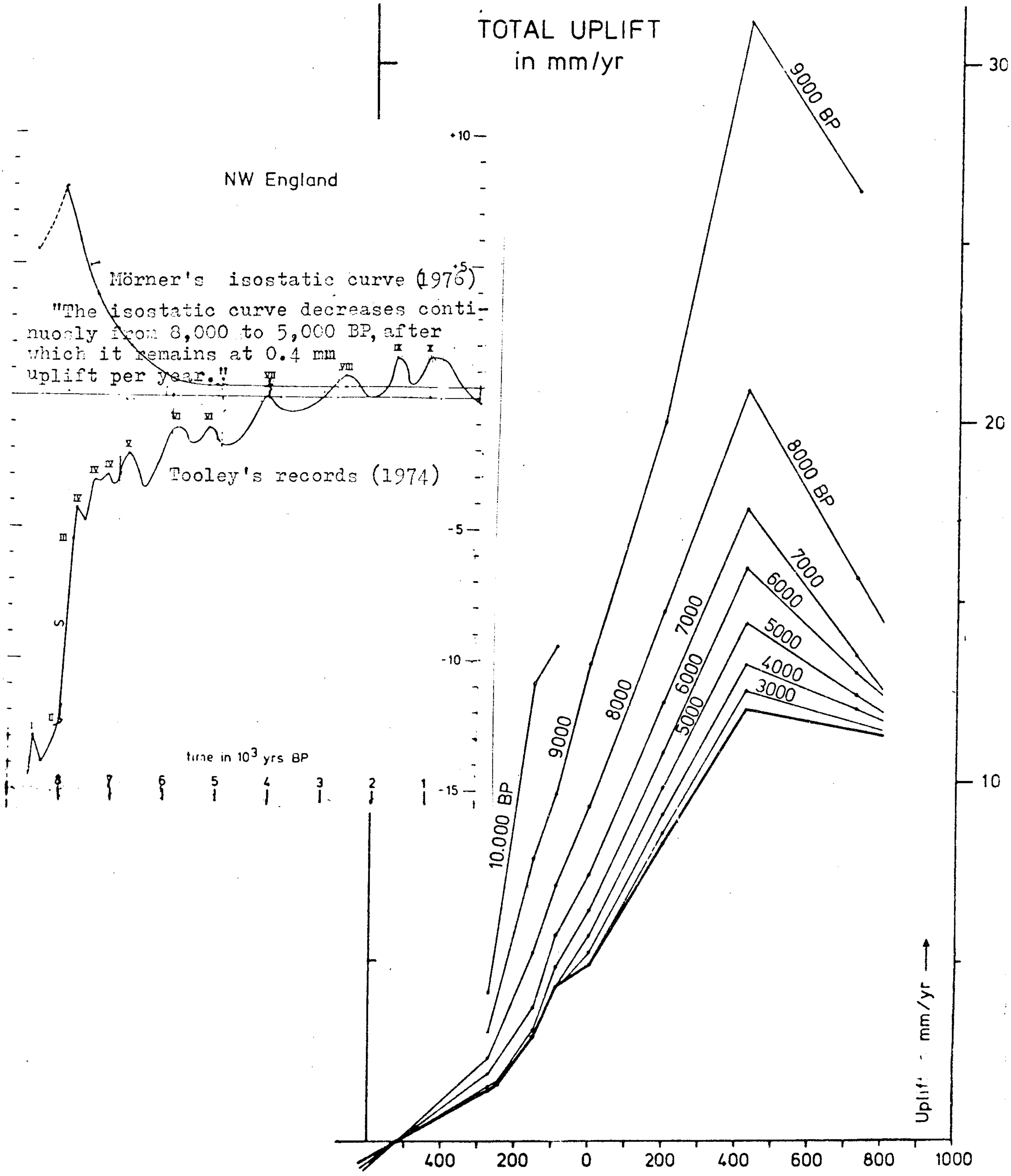

"Total uplift in mm/year for every 1000 yaer BP along the East Coast profile. Thick line gives the present rate of uplift. (Geodetic!) Horizontal scale in km with Stockholm as zero point. The graph demonstrates that the Swedish uplift is complex and composed of two factors; one glacio-isostatic factor that decreased continuosly with time and distance from the periphery and died out some 2000-3000 years BP, and one "tectonic" factor that has remained constant and is responsible for the present uplift." After N.A. Mörner (1976).

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