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THERMOELASTIC STRESS DUE TO A RECTANGULAR HEAT SOURCE IN A SEMI-INFINITE MEDIUM.

1. Derivation of an Analytical Solution.

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Contents

Abstract

The thermoelastic response due to a time-dependent rectangular heat source in a semi-infinite medium is analysed. The problem originates from studies of nuclear waste repositories in rock. Canisters containing nuclear waste are deposited over a large rectangular area deep below the ground surface. An important concern is that dangerous waste from damaged canisters may eventually reach the biosphere by groundwater moving in cracks and fractures in the rock. The stress and strain fields are therefore of main interest, since they influence crack formation and crack widths.

The solution for a time-dependent heat source is obtained from the corresponding instantaneous heat source by a Duhamel superposition. The thermoelastic problem for the instantaneous rectangular heat source in an infinite surrounding is solved exactly. An important step is the introduction of so called quadrantal heat sources. The solution for the rectangle is obtained from four quadrantal solutions. The solution for the quadrantal heat source depends on the three dimensionless coordinates only. Time occurs in the scale factors only.

The condition of zero normal and shear stresses at the ground surface is fulfilled by using a mirror heat source and a boundary solution. The boundary solution accounts for the residual normal stress at the ground surface. Using a Hertzian potential, a surprisingly simple solution is obtained.

The final analytical solution is quite tractable, considering the complexity of the initial problem. The solution may be used to test numerical models for coupled thermoelastic processes. It may also be used in more detailed numerical simulations of the process near the heat sources as boundary conditions to account for the three-dimensional global process.

Sammanfattning

Den termoelastiska responsen på en tidsberoende rektangulär värmekälla i ett halvoändligt medium analyseras. Problemet härrör från studier av kärnbränslelager i berg. Kapslar med radioaktivt avfall placeras över en stor rektangulär area djupt nere under markytan. Farligt avfall från skadade kapslar med kärnbränsle kan potentiellt nå biosfären med grundvatten som strömmar i sprickor i berget. Töjnings- och spänningsfälten är därför av intresse eftersom dessa påverkar spickbildning och sprickvidd.

Lösningen för en tidsberoende värmekälla erhålls från motsvarande momentana värmekälla genom superposition. Det termoelastiska problemet för en momentan rektangulär värmekälla i en oändlig omgivning löses exakt. Ett viktigt steg är införandet av en så kallad kvadrantvärmekälla. Lösningen för rektangeln erhålls från fyra kvadrantlösningar. Lösningen för kvadrantvärmekällan beror enbart på de tre dimensionslösa koordinaterna. Tiden uppträder enbart i skalfaktorer.

Villkoret att normal- och skjuvspänningar är noll vid markytan uppfylls med hjälp av en spegelvärmekälla och en 'randlösning'. Denna lösning tar hand om den resterande normalspänningen vid markytan. Med hjälp av en Hertzsk potential erhålles en anmärkingsvärt enkel lösning.

Den slutliga lösningen är förhållandevis hanterlig om man beaktar det ursprungliga problemets komplexitet. Lösningen kan användas för att testa numeriska modeller av kopplade termoelastiska processer. Den kan också utnyttjas i mer detaljerade numeriska simuleringar av förloppet nära värmakällorna som randvillkor för att få med den mer globala tredimensionella processen.

1 Introduction

In the KBS-3 concept, nuclear waste from the Swedish nuclear power plants is put in some six thousand canisters, which are buried in solid rock at a depth H of 500 m below the ground surface. See Figure 1, left. The canisters are placed in boreholes below parallel tunnels at a spacing D of about 6 m. The distance D' between the tunnels is about 25 m. The nuclear waste repository consists of canisters in a large, rectangular grid. The total area of the rectangle with the side lengths 2L and 2B is almost 1 km^2 ($6{\cdot}25 \cdot 6000$ m^2). Each canister lies at the center of a small rectangle with the side lengths D and D'.

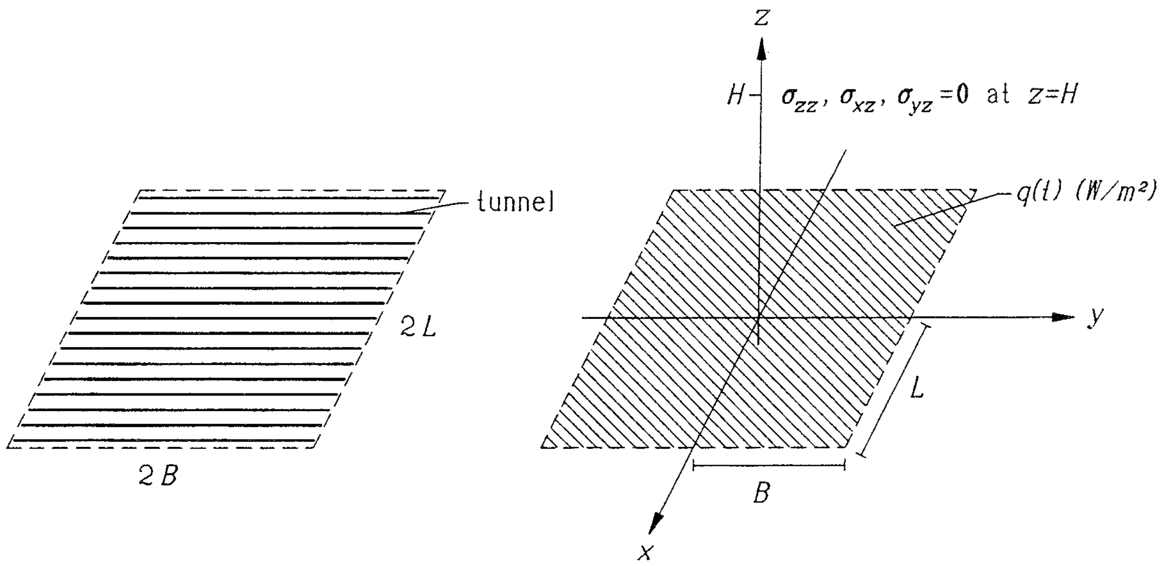

Figure 1: Left: Nuclear waste repository with heat-emitting canisters placed along tunnels. Right: Thermoelastic problem with a time-dependent rectangular heat source plane. The stress is zero at the ground surface $z = 0$.

The canisters emit heat due to radioactive decay in the nuclear waste. The heat sources from all canisters create a complex three-dimensional, time-dependent temperature field in the ground in and around the repository. This thermal problem is studied in Claesson, Probert (Jan. 1996). The local problem around a particular canister is not considered here. On a larger scale (above D and D'), there is a rectangular heat source plane:

$$q(t) \quad \mathrm{W/m^2} \quad \text{at} \quad -L < x < L, \; -B < y < B \;\; z = 0 \tag{1}$$

The heat emission per canister becomes $DD'q(t)$ (W). The heat emission decreases with time in a known way. We will use a sum of a few exponentials with different decay times to represent the function $q(t)$. The emitted heat warms the rock and induces a thermoelastic stress field. The rock mass serves as a protective barrier against the nuclear waste. In the worst-case scenario, groundwater may transport nuclear waste all the way from damaged canisters to the biosphere. Groundwater flow requires an open fracture and crack system. The stress and strain fields are therefore of main interest, since they influence crack closure, opening, formation and widths.

Thermal stresses and temperature fields are studied in Israelsson (1995). Solutions for point sources from all canisters are summed directly.

The purpose of this study is to analyse the thermoelastic process in the rock caused by the rectangular heat source. The process is of interest for different time-scales from the first years to thousands of years. The behavior in and around the repository region, but also far away from the canisters, is of interest.

A particular aim for the analytical approach is to gain a physical understanding and the possibility to quantify particular processes and their interactions.

An exact analytical solution for the time-dependent, three-dimensional process is derived. The solution, which is not valid in the immediate vicinity of single canisters with their local complications, is surprisingly simple, considering the complexity of the coupled process.

The solution may be used as boundary conditions in numerical modelling of the local processes around a canister. There are not many analytical solutions for more complex coupled thermo-elastic processes. The presented solution should be a good one to test numerical models.

General references for thermo-elasticity is Boley and Weiner (1960) and Parkus (1959). General formulas used here are taken from the latter reference. The radial problem for a point heat source may be found in textbooks, for example Timoshenko and Goodier (1970). In our literature search, we did not find any solution for the rectangular heat source.

This paper deals mainly with derivation of the solution with its various components. The results will be applied to the KBS-3 case in a second report. The corresponding problem of the thermoelastic response to a single, finite line heat source is studied in Claesson, Hellström (1995).

2 Mathematical problem

The linearly elastic, isotropic, homogeneous medium is semi-infinite: $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < H$. In the basic case, the thermoelastic stress and strain fields are caused by the temperature field $T(x, y, z, t)$ of an instantaneous heat source. The heat e_0 (J/m^2) is emitted at $t = 0$ over a rectangular surface $z = 0$, $-L < x < L$, $-B < y < B$.

The solution for any time-dependent heat source $q(t)$ (W/m^2) is then obtained by superposition using a Duhamel integral. See Section 20. We will in particular consider the case when $q(t)$ consists of a sum of exponentially decaying components. The solution for this case is studied in Section 21.

The displacement field $\mathbf{u} = (u, v, w)$ satisfies Navier's equations, Parkus (1959):

$$\nabla^2(\mathbf{u}) + \frac{1}{1-2\nu}\nabla(e) = \frac{2\alpha(1+\nu)}{1-2\nu}\nabla T \qquad e = \nabla \cdot \mathbf{u} \tag{2}$$

Here, ν (-) is Poisson's ratio, e (-) the volume expansion and α (1/K) the coefficient of linear thermal expansion. The strain components are:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \ldots \tag{3}$$

The relations between strain and stress are, Parkus (1959):

$$\sigma_{xx} = 2\mu\left(\varepsilon_{xx} + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T\right) \quad \ldots$$

$$\sigma_{xy} = 2\mu\varepsilon_{xy} \quad \ldots \qquad \mu = \frac{E}{2(1+\nu)} \tag{4}$$

Here, E denotes Young's modulus and μ the modulus of shear.

The solution $(u, \varepsilon_{xx}, \sigma_{xx} \ldots)$ shall tend to zero at infinity ($\sqrt{x^2+y^2} \to \infty$ or $z \to -\infty$). The stress components at the ground surface $z = H$ are zero:

$$\sigma_{zz} = 0 \quad \sigma_{zx} = 0 \quad \sigma_{zy} = 0 \quad \text{at} \quad z = H \tag{5}$$

3 Temperature field

The heat e_0 (J/m^2) is emitted at $t = 0$ over the rectangular surface $-L < x < L$, $-B < y < B$, $z = 0$. The total emitted heat becomes $4BLe_0$ (J). The temperature field due to this heat source involves the two lengths L and B. A similar problem without these lengths is to consider the heat source:

$$e_0 \cdot \text{sign}(x) \cdot \text{sign}(y) \quad \text{over} \quad z = 0 \quad \text{at} \quad t = 0 \tag{6}$$

The heat e_0 (J/m^2) is emitted in the quadrants $x > 0$, $y > 0$, $z = 0$ and $x < 0$, $y < 0$, $z = 0$, while $-e_0$ (J/m^2) is emitted in the two other quadrants $x > 0$, $y < 0$, $z = 0$ and $x < 0$, $y > 0$, $z = 0$. We will call this heat source distribution a *quadrantal* heat source. Let $T_{qi}(x, y, z, t)$ be the temperature field from the quadrantal heat source in infinite space $-\infty < x, y, z < \infty$. This temperature is superimposed on the undisturbed temperature field in the ground. We have, Carslaw-Jaeger (1959):

$$T_{qi}(x,y,z,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{e_0 \cdot \text{sign}(x') \cdot \text{sign}(y')}{\rho c (4\pi a t)^{3/2}} \cdot \mathrm{e}^{-[(x-x')^2+(y-y')^2+z^2]/(4at)} \tag{7}$$

or

$$T_{qi}(x,y,z,t) = \frac{e_0}{\rho c} \cdot \text{erf}\left(\frac{x}{\sqrt{4at}}\right) \cdot \text{erf}\left(\frac{y}{\sqrt{4at}}\right) \cdot \frac{1}{\sqrt{4\pi a t}} \cdot \mathrm{e}^{-z^2/(4at)} \tag{8}$$

Here, a (m^2/s) denotes the thermal diffusivity, ρ (kg/m^3) the density, c (J/(kg·K)) the specific heat capacity, and $\text{erf}(x)$ the error function:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-s^2} ds \tag{9}$$

The temperature field $T_{qi}(x, y, z, t)$ involves integrals in x and y of the field from point heat sources. The second-order derivative with respect to x and y gives the field from a point source at the center $(0, 0, 0)$. We have:

$$\frac{\partial^2 T_{qi}}{\partial x \partial y} = \frac{4e_0}{\rho c} \cdot \frac{1}{\sqrt{4\pi a t}^3} \cdot \mathrm{e}^{-(x^2+y^2+z^2)/(4at)} \tag{10}$$

This fact will be used in Section 6.

The temperature field from the rectangular heat source is obtained from superposition of four quadrantal solutions of the above type. This follows from the identity:

$$0.5\,[\text{sign}(x+L) - \text{sign}(x-L)] \cdot 0.5\,[\text{sign}(y+B) - \text{sign}(y-B)] =$$

$$= \begin{cases} 1 & |x| < L \quad \text{and} \quad |y| < B \\ 0 & \text{outside the rectangle} \end{cases} \tag{11}$$

The first factor in x is equal to $+1$ for $|x| < L$ and 0 for $|x| > L$. The other factor has the same behaviour in y. The product is equal to $+1$ for $|x| < L$ and $|y| < B$, and it is 0 elsewhere. We get four products of the type $(\pm)\text{sign}(x \pm L) \cdot (\pm)\text{sign}(y \pm B)$. Each of these corresponds to a solution T_{qi} with x replaced by $x \pm L$ and y by $y \pm B$. The temperature field from the instantaneous rectangular heat source is then:

$$\begin{aligned} T_{rec,i}(x,y,z,t) \;=\; & 0.25 \cdot [T_{qi}(x+L, y+B, z, t) - T_{qi}(x+L, y-B, z, t,) \\ & - T_{qi}(x-L, y+B, z, t) + T_{qi}(x-L, y-B, z, t)] \end{aligned} \tag{12}$$

Here, T_{qi} is given by (8). The above temperature field can be rewritten in the following more compact form:

$$T_{rec,i}(x,y,z,t) = \sum_{n_x=\pm 1} \sum_{n_y=\pm 1} \frac{n_x n_y}{4} \cdot T_{qi}(x + n_x L, y + n_y B, z, t) \tag{13}$$

The summation indices n_x and n_y assume the values +1 and −1 only.

The solution for any time-varying heat source $q(t)$ over the rectangle is obtained by superposition as a Duhamel integral, Carslaw-Jaeger (1959) and Subsection 20.1:

$$T(x,y,z,t) = \sum_{n_x=\pm 1} \sum_{n_y=\pm 1} \frac{n_x n_y}{4} \cdot \int_0^t \frac{q(t')}{e_0} \cdot T_{qi}(x + n_x L, y + n_y B, z, t - t')dt' \tag{14}$$

The factor e_0 cancels in the above formula, since t_{qi} is proportional to e_0.

The essential problem to solve is the thermoelastic response to the quadrantal temperature field $T_{qi}(x,y,z,t)$. The response for the rectangular heat source is obtained by superposition of four solutions with x replaced by $x \pm L$ and y by $y \pm B$ as in Eq. (13). The solution for any $q(t)$ is then obtained by a Duhamel integral as in Eq. (14).

4 Displacement potential for infinite space

We first consider the problem for infinite space ($-\infty < z < \infty$) without the boundary conditions (5) at the ground surface $z = H$. A second solution to account for the boundary is studied in Section 12 and onwards.

A few thermoelastic problems may be solved by use of a single displacement potential $\Phi(x,y,z,t)$ (m^2), Parkus (1959):

$$\mathbf{u} = \nabla\Phi \tag{15}$$

This is an equation of Poisson's type. Navier's equation (2) is satisfied if Φ is a solution of

$$\nabla^2\Phi = \frac{1+\nu}{1-\nu}\alpha T \tag{16}$$

The temperature field is considered at any time $t > 0$, so $\Phi(x,y,z,t)$ depends on the spacial coordinates with t as a parameter.

Eq. (16) is to be solved for $T = T_{qi}$. The displacement potential is the solution of:

$$\nabla^2\Phi = \frac{1+\nu}{1-\nu} \cdot \frac{e_0\alpha}{\rho c\sqrt{\pi}} \cdot \frac{1}{\sqrt{4at}} \cdot \mathrm{erf}\left(\frac{x}{\sqrt{4at}}\right) \cdot \mathrm{erf}\left(\frac{y}{\sqrt{4at}}\right) \cdot e^{-z^2/(4at)} \tag{17}$$

The temperature T_{qi} on the right hand side is antisymmetric in x and y. We will choose a particular solution which is odd in x and in y.

The introduction of the quadrantal heat source is *a crucial step* to facilitate the analysis. The lengths L and B do not occur in the quadrantal problem. The problem (for infinite space) involves the spacial coordinates only. The time t occurs as the factor $\sqrt{4at}$. We will see in the dimensionless formulation in the next section that time only occurs as a scale factor. The basic quadrantal solution depends on the three dimensionless coordinates only.

The spacial derivatives of $\Phi(x,y,z,t)$ give the three displacement components:

$$u = \frac{\partial\Phi}{\partial x} \qquad v = \frac{\partial\Phi}{\partial y} \qquad w = \frac{\partial\Phi}{\partial z} \tag{18}$$

The second-order derivatives give the strain components, Eq. (3):

$$\varepsilon_{xx} = \frac{\partial^2 \Phi}{\partial x^2} \qquad \varepsilon_{yy} = \frac{\partial^2 \Phi}{\partial y^2} \qquad \varepsilon_{zz} = \frac{\partial^2 \Phi}{\partial z^2}$$

$$\varepsilon_{xy} = \frac{\partial^2 \Phi}{\partial x \partial y} \qquad \varepsilon_{xz} = \frac{\partial^2 \Phi}{\partial x \partial z} \qquad \varepsilon_{yz} = \frac{\partial^2 \Phi}{\partial y \partial z} \tag{19}$$

The volume expansion is given by the Laplacian of Φ:

$$e = \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{20}$$

The stress components are given by the following simple expressions, Parkus 1959:

$$\sigma_{xx} = 2\mu \left(\frac{\partial^2 \Phi}{\partial x^2} - \nabla^2 \Phi \right) \qquad \sigma_{xy} = 2\mu \frac{\partial^2 \Phi}{\partial x \partial y}$$

$$\sigma_{yy} = 2\mu \left(\frac{\partial^2 \Phi}{\partial y^2} - \nabla^2 \Phi \right) \qquad \sigma_{xz} = 2\mu \frac{\partial^2 \Phi}{\partial x \partial z}$$

$$\sigma_{zz} = 2\mu \left(\frac{\partial \Phi}{\partial z^2} - \nabla^2 \Phi \right) \qquad \sigma_{yz} = 2\mu \frac{\partial^2 \Phi}{\partial y \partial z} \tag{21}$$

5 Dimensionless formulation

The length $\sqrt{4at}$ is a measure of the thermal influence range from the heat source at time t. This length, which is the only one occurring in the problem (17) except for the coordinates x, y and z, is used for the dimensionless coordinates:

$$x' = \frac{x}{\sqrt{4at}} \qquad y' = \frac{y}{\sqrt{4at}} \qquad z' = \frac{z}{\sqrt{4at}} \tag{22}$$

The dimensionless gradient and Laplace operators become:

$$\nabla = \frac{1}{\sqrt{4at}} \nabla' \qquad \nabla^2 = \frac{1}{4at} (\nabla')^2 \tag{23}$$

The displacement potential Φ satisfies Eq. (17). Inserting the dimensionless Laplace operator, we get:

$$\frac{1}{4at} \cdot (\nabla')^2 \Phi = \frac{1+\nu}{1-\nu} \cdot \frac{e_0 \alpha}{\pi \rho c} \cdot \frac{1}{\sqrt{4at}} \cdot T_q(x', y', z') \tag{24}$$

Here, we have introduced the dimensionless temperature T_q:

$$T_q(x', y', z') = \sqrt{\pi} \cdot \text{erf}(x') \cdot \text{erf}(y') \cdot e^{-(z')^2} \tag{25}$$

All quantities with index q refer to the antisymmetric quadrantal temperature field of Eq. (25). They are dimensionless and depend only on the dimensionless coordinates x', y' and z'. The original temperature T_{qi}, Eq. (8), is:

$$T_{qi}(x, y, z, t) = \frac{e_0}{\pi \rho c} \cdot \frac{1}{\sqrt{4at}} \cdot T_q(x', y', z') \tag{26}$$

A dimensionless displacement potential is defined by:

$$\Phi(x, y, z, t) = u_0 \sqrt{4at} \cdot \Phi_q(x', y', z') \tag{27}$$

Here, u_0 (m) is defined by:

$$u_0 = \frac{1+\nu}{1-\nu} \cdot \frac{e_0\alpha}{\pi\rho c} \tag{28}$$

Then from Eqs. (24), (27) and (28), $\Phi_q(x', y', z')$ shall satisfy:

$$(\nabla')^2\Phi_q = T_q \tag{29}$$

Here, $T_q(x', y', z')$ is given by Eq. (25).

The displacement vector $\mathbf{u}$ becomes, Eqs. (15), (23) and (27):

$$\mathbf{u} = \frac{1}{\sqrt{4at}}\nabla'\left(u_0\sqrt{4at} \cdot \Phi_q\right) = u_0 \cdot \mathbf{u}_q \tag{30}$$

The constant u_0, Eq. (28), is the scale factor for the displacement. The components of the dimensionless displacement $\mathbf{u}_q$ are:

$$u^q = \frac{\partial\Phi_q}{\partial x'} \qquad v^q = \frac{\partial\Phi_q}{\partial y'} \qquad w^q = \frac{\partial\Phi_q}{\partial z'} \tag{31}$$

(The index q is placed in the upper position for all components of displacement, stress and strain.)

The strain components become, Eqs. (19), (22) and (27):

$$\varepsilon_{xx} = \frac{u_0}{\sqrt{4at}} \cdot \varepsilon^q_{xx} \qquad \varepsilon_{xy} = \frac{u_0}{\sqrt{4at}} \cdot \varepsilon^q_{xy} \qquad \ldots \tag{32}$$

The scale factor for strain becomes $u_0/\sqrt{4at}$. The dimensionless strain components, all with an upper index q, are given by second-order derivatives of Φ_q:

$$\varepsilon^q_{xx} = \frac{\partial^2\Phi_q}{\partial(x')^2} \qquad \varepsilon^q_{xy} = \frac{\partial^2\Phi_q}{\partial x'\partial y'} \qquad \ldots \tag{33}$$

The stress components become, Eqs. (20-21):

$$\sigma_{xx} = \frac{p_0}{\sqrt{4at}} \cdot \sigma^q_{xx} \qquad \sigma_{xy} = \frac{p_0}{\sqrt{4at}} \cdot \sigma^q_{xy} \qquad \ldots \tag{34}$$

The scale factor for stress is $p_0/\sqrt{4at}$. The constant p_0 (Pa·m) is, Eq. (28):

$$p_0 = 2\mu u_0 = \frac{E}{1+\nu} \cdot u_0 = \frac{E}{1-\nu} \cdot \frac{e_0\alpha}{\pi\rho c} \tag{35}$$

The dimensionless stress components are given by:

$$\sigma^q_{xx} = \frac{\partial^2\Phi_q}{\partial(x')^2} - (\nabla')^2\Phi_q \qquad \sigma^q_{xy} = \frac{\partial^2\Phi_q}{\partial x'\partial y'} \qquad \ldots \tag{36}$$

6 Calculation of displacement potential

The dimensionless displacement potential $\Phi_q(x', y', z')$ is the solution of Eqs. (29) and (25):

$$(\nabla')\Phi_q = \sqrt{\pi} \cdot \mathrm{erf}(x') \cdot \mathrm{erf}(y') \cdot \mathrm{e}^{-(z')^2} \tag{37}$$

In order to obtain a solution, the derivatives with respect to x' and to y' are performed, Eq. (10):

$$(\nabla')^2\left[\frac{\partial^2\Phi_q}{\partial x'\partial y'}\right] = \sqrt{\pi} \cdot \frac{2}{\sqrt{\pi}}\mathrm{e}^{-(x')^2} \cdot \frac{2}{\sqrt{\pi}}\mathrm{e}^{-(y')^2} \cdot \mathrm{e}^{-(z')^2} = \frac{4}{\sqrt{\pi}}\mathrm{e}^{-(r')^2} \tag{38}$$

Here, r' is the radial distance. The right-hand side depends on r' only. The Laplace operator with radial variation only gives:

$$\frac{1}{r'}\frac{d^2}{d(r')^2}\left[r'\cdot\frac{\partial^2\Phi_q}{\partial x'\partial y'}\right] = \frac{4}{\sqrt{\pi}}e^{-(r')^2} \tag{39}$$

Two integrations in r' give:

$$\frac{\partial^2\Phi_q}{\partial x'\partial y'} = -\frac{\text{erf}(r')}{r'} + A_0 + \frac{A_1}{r'} \tag{40}$$

The solution is regular at $r' = 0$. This means that $A_1 = 0$, since $\text{erf}(r')/r'$ is regular at $r' = 0$.

The constant A_0 is put to zero. We had the requirement that Φ_q is odd in x' and in y', which means that Φ_q is zero for $x' = 0$ and $y' = 0$. Then we get by integration in x' and in y' the following solution:

$$\Phi_q(x', y', z') = -\int_0^{x'} dx'' \int_0^{y'} dy'' \cdot \frac{\text{erf}\left(\sqrt{(x'')^2+(y'')^2+(z')^2}\right)}{\sqrt{(x'')^2+(y'')^2+(z')^2}} \tag{41}$$

This is our basic solution for the problem in infinite space. The strain and stress components are obtained from second-order derivatives, Eqs. (33) and (36).

The double-integral (41) for Φ_q may be written in a quite different form. The factor erf(...) is given by an integral in s from 0 to $f = \sqrt{(x'')^2+(y'')^2+(z')^2}$, Eq. (9). The substitution $s = s'f$ is made. The factor f cancels. Integrations in x'' and in y'' are readily performed. The following alternative expression for Φ_q is obtained:

$$\Phi_q(x', y', z') = -\frac{\sqrt{\pi}}{2}\int_0^1 \frac{\text{erf}(x's)\cdot\text{erf}(y's)}{s^2}e^{-(z')^2s^2}ds \tag{42}$$

Here, Φ_a is given by a single integral. The integrand is regular at $s = 0$, since $\text{erf}(p)$ is proportional to p near $p = 0$.

7 Displacement field

The dimensionless displacement field $\mathbf{u}_q$ is obtained by the first-order derivatives of $\Phi_q(x', y', z')$. From Eq. (41), we get:

$$u^q = \frac{\partial\Phi_q}{\partial x'} = -\int_0^{y'} \frac{\text{erf}\left(\sqrt{(x')^2+(y'')^2+(z')^2}\right)}{\sqrt{(x')^2+(y'')^2+(z')^2}}dy'' \tag{43}$$

$$v^q = \frac{\partial\Phi_q}{\partial y'} = -\int_0^{x'} \frac{\text{erf}\left(\sqrt{(x'')^2+(y')^2+(z')^2}\right)}{\sqrt{(x'')^2+(y')^2+(z')^2}}dx'' \tag{44}$$

The expression for w^q involves a double-integral. These integrals have to be evaluated numerically.

Alternative expressions are obtained from derivatives of Eq. (42):

$$u^q = -\int_0^1 \frac{\text{erf}(y's)}{s}\cdot e^{-[(x')^2+(z')^2]s^2}ds$$

$$v^q = -\int_0^1 \frac{\text{erf}(x's)}{s}\cdot e^{-[(y')^2+(z')^2]s^2}ds$$

$$w^q = \sqrt{\pi} z' \int_0^1 \text{erf}(x's) \cdot \text{erf}(y's) \cdot e^{-(z')^2 s^2} ds \tag{45}$$

The dimensionless displacement $\mathbf{u}_q$ multiplied by u_0 gives the real displacement $\mathbf{u}$, Eq. (30). We have after the substitution $s/\sqrt{4at} = s'$ in the integrals above:

$$u^{qi}(x, y, z, t) = -u_0 \int_0^{1/\sqrt{4at}} \frac{\text{erf}(ys)}{s} \cdot e^{-(r^2-y^2)s^2} ds$$

$$v^{qi}(x, y, z, t) = -u_0 \int_0^{1/\sqrt{4at}} \frac{\text{erf}(xs)}{s} \cdot e^{-(r^2-x^2)s^2} ds$$

$$w^{qi}(x, y, z, t) = u_0 \sqrt{\pi} z \int_0^{1/\sqrt{4at}} \text{erf}(xs) \cdot \text{erf}(ys) \cdot e^{-z^2 s^2} ds \tag{46}$$

The index qi is introduced to denote the instantaneous *quadrantal* solution for an *infinite* medium. The integrals are quite simple to evaluate numerically.

8 Second-order derivatives

The stress and strain fields are given by the second-order derivatives of $\Phi_q(x', y', z')$. We have:

$$\frac{\partial^2 \Phi_q}{\partial (x')^2} = \frac{1}{r'} \cdot \frac{x'y'}{(x')^2 + (z')^2} \left[\text{erf}(r') - r' e^{-(x')^2-(z')^2} \cdot \frac{\text{erf}(y')}{y'} \right]$$

$$\frac{\partial^2 \Phi_q}{\partial (y')^2} = \frac{1}{r'} \cdot \frac{x'y'}{(y')^2 + (z')^2} \left[\text{erf}(r') - r' e^{-(y')^2-(z')^2} \cdot \frac{\text{erf}(x')}{x'} \right]$$

$$\frac{\partial^2 \Phi_q}{\partial (z')^2} = T_q - \frac{\partial^2 \Phi_q}{\partial (x')^2} - \frac{\partial^2 \Phi_q}{\partial (y')^2}$$

$$\frac{\partial^2 \Phi_q}{\partial x' \partial y'} = -\frac{1}{r'} \cdot \text{erf}(r')$$

$$\frac{\partial^2 \Phi_q}{\partial x' \partial z'} = \frac{1}{r'} \cdot \frac{y'z'}{(x')^2 + (z')^2} \left[\text{erf}(r') - r' e^{-(x')^2-(z')^2} \cdot \frac{\text{erf}(y')}{y'} \right]$$

$$\frac{\partial^2 \Phi_q}{\partial y' \partial z'} = \frac{1}{r'} \cdot \frac{x'z'}{(y')^2 + (z')^2} \left[\text{erf}(r') - r' e^{-(y')^2-(z')^2} \cdot \frac{\text{erf}(x')}{x'} \right] \tag{47}$$

The first formula for the second-order derivative with respect to x' is obtained from Eq. (43) or (45). A partial integration has to be performed in order to remove the integral. The second formula is obtained in the same way. The second-order derivative with respect to z' is more difficult. Therefore, Eq. (29) is used. The fourth formula is obtained immediately from Eq. (41). The fifth formula is obtained from Eq. (44) or Eq. (45). Here, a partial integration is again necessary. The last formula is obtained in the same way.

The same factor occurs in the brackets of the second and sixth formula of (47). It is a function of x' and r':

$$A'(x', r') = \text{erf}(r') - r' e^{-[(r')^2-(x')^2]} \cdot \frac{\text{erf}(x')}{x'} \qquad (r')^2 - (x')^2 = (y')^2 + (z')^2 \tag{48}$$

The expression within brackets in the first and fifth formula of (47) is equal to $A'(y', r')$.

From Eqs. (33), (36), (29), (47) and (48), the dimensionless strain and stress components are:

$$\varepsilon^q_{xx} = \sigma^q_{xx} + T_q = \frac{1}{r'} \cdot \frac{x'y'}{(x')^2 + (z')^2} \cdot A'(y', r')$$

$$\varepsilon^q_{yy} = \sigma^q_{yy} + T_q = \frac{1}{r'} \cdot \frac{x'y'}{(y')^2 + (z')^2} \cdot A'(x', r')$$

$$\varepsilon^q_{zz} = \sigma^q_{zz} + T_q = T_q - \frac{x'y'}{r'} \left[\frac{A'(y', r')}{(x')^2 + (z')^2} + \frac{A'(x', r')}{(y')^2 + (z')^2} \right]$$

$$\varepsilon^q_{xy} = \sigma^q_{xy} = -\frac{1}{r'} \cdot \mathrm{erf}(r')$$

$$\varepsilon^q_{xz} = \sigma^q_{xz} = \frac{1}{r'} \cdot \frac{y'z'}{(x')^2 + (z')^2} \cdot A'(y', r')$$

$$\varepsilon^q_{yz} = \sigma^q_{yz} = \frac{1}{r'} \cdot \frac{x'z'}{(y')^2 + (z')^2} \cdot A'(x', r') \qquad (49)$$

Here, T_q is given by (25) and A' by (48).

9 Strain field

The real components of the strain field are obtained from the dimensionless expressions above, Eqs. (49), by insertion of the real coordinates $x = \sqrt{4at} \cdot x'$ etc., Eq. (22), and multiplication by the scale factor $u_0/\sqrt{4at}$, Eq. (32). We get:

$$\varepsilon^{qi}_{xx} = \frac{u_0}{r} \cdot \frac{xy}{x^2 + z^2} \cdot A(y, r, t) \qquad \varepsilon^{qi}_{yy} = \frac{u_0}{r} \cdot \frac{xy}{y^2 + z^2} \cdot A(x, r, t)$$

$$\varepsilon^{qi}_{zz} = \frac{1 + \nu}{1 - \nu} \alpha \cdot T_{qi}(x, y, z, t) - \frac{u_0}{r} \left[\frac{xy}{x^2 + z^2} \cdot A(y, r, t) + \frac{xy}{y^2 + z^2} \cdot A(x, r, t) \right]$$

$$\varepsilon^{qi}_{xz} = \frac{u_0}{r} \cdot \frac{yz}{x^2 + z^2} \cdot A(y, r, t) \qquad \varepsilon^{qi}_{yz} = \frac{u_0}{r} \cdot \frac{xz}{y^2 + z^2} \cdot A(x, r, t)$$

$$\varepsilon^{qi}_{xy} = -\frac{u_0}{r} \cdot \mathrm{erf}\left(\frac{r}{\sqrt{4at}} \right) \qquad u_0 = \frac{1 + \nu}{1 - \nu} \cdot \frac{e_0 \alpha}{\pi \rho c} \qquad (50)$$

The upper index qi refers to the quadrantal solution in an infinite medium. The temperature field T_{qi} is defined by Eq. (26) or (8). The auxiliary function A is given by, Eq. (48):

$$A(p, r, t) = A'(p', r') = A'\left(\frac{p}{\sqrt{4at}}, \frac{r}{\sqrt{4at}} \right) =$$

$$= \mathrm{erf}\left(\frac{r}{\sqrt{4at}} \right) - r \cdot e^{-(r^2 - p^2)/(4at)} \cdot \frac{\mathrm{erf}(p/\sqrt{4at})}{p} \qquad p = x,\ y \qquad (51)$$

10 Stress field

The components of the real stress field are obtained from the dimensionless equations (49) by multiplication with the scale factor $p_0/\sqrt{4at}$, (35):

$$\sigma^{qi}_{xx} = \frac{p_0}{r} \cdot \frac{xy}{x^2 + z^2} \cdot A(y, r, t) - \frac{E\alpha}{1 - \nu} \cdot T_{qi}(x, y, z, t)$$

$$\sigma^{qi}_{yy} = \frac{p_0}{r}\cdot\frac{xy}{y^2+z^2}\cdot A(x,r,t) - \frac{E\alpha}{1-\nu}\cdot T_{qi}(x,y,z,t)$$

$$\sigma^{qi}_{zz} = -\frac{p_0}{r}\left[\frac{xy}{x^2+z^2}\cdot A(y,r,t) + \frac{xy}{y^2+z^2}\cdot A(x,r,t)\right]$$

$$\sigma^{qi}_{xz} = \frac{p_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot A(y,r,t) \qquad \sigma^{qi}_{yz} = \frac{p_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot A(x,r,t)$$

$$\sigma^{qi}_{xy} = -\frac{p_0}{r}\cdot \operatorname{erf}\left(\frac{r}{\sqrt{4at}}\right) \qquad p_0 = \frac{E}{1+\nu}\cdot u_0 = \frac{E}{1-\nu}\cdot\frac{e_0\alpha}{\pi\rho c} \tag{52}$$

The functions A and T_{qi} are defined Eqs. (51) and (8).

The above formulas in Sections by 9 and 10 give the complete analytical solution for the strain and stress fields caused by the instantaneous quadrantal heat source (6) with the temperature (8) in infinite space.

11 Far-field

The behaviour far away from the heat source plane $z = 0$ is of particular interest. Inserting $z' \gg 1$ in Eq. (41), we have:

$$\operatorname{erf}\left(\sqrt{\ldots + (z')^2}\right) \simeq 1 \qquad (z' \gg 1)$$

$$\Phi_q(x',y',z') \simeq -\int_0^{x'} dx'' \int_0^{y'} dy'' \frac{1}{\sqrt{(x'')^2+(y'')^2+(z')^2}} \tag{53}$$

The limit $z' > 2$ is sufficient in our application, since erf(2)=0.995. The higher limit $z' > 3$ will always be an extremely good approximation, since erf(3)=0.99998.

The double integral may (with some difficulty) be evaluated analytically. The result is:

$$\Phi_q(x',y',z') \simeq -\left[\frac{x'}{2}\ln\left(\frac{r'+y'}{r'-y'}\right) + \frac{y'}{2}\ln\left(\frac{r'+x'}{r'-x'}\right) - z'\cdot\arctan\left(\frac{x'y'}{z'r'}\right)\right]$$

$$r' = \sqrt{(x')^2+(y')^2+(z')^2} \qquad (z' > 2) \tag{54}$$

In real coordinates, we get from (27):

$$\Phi(x,y,z) \simeq -u_0\left[\frac{x}{2}\ln\left(\frac{r+y}{r-y}\right) + \frac{y}{2}\ln\left(\frac{r+x}{r-x}\right) - z\cdot\arctan\left(\frac{xy}{zr}\right)\right] \tag{55}$$

The expression for the far-field potential satisfies Laplace equation $\nabla^2\Phi = 0$.

The first-order derivatives give the displacement components:

$$u^{qi} \simeq -\frac{u_0}{2}\ln\left(\frac{r+y}{r-y}\right) \qquad v^{qi} \simeq -\frac{u_0}{2}\ln\left(\frac{r+x}{r-x}\right) \qquad w^{qi} \simeq u_0\cdot\arctan\left(\frac{xy}{zr}\right)$$

$$r = \sqrt{x^2+y^2+z^2} \qquad (z' = \frac{z}{\sqrt{4at}} > 2) \tag{56}$$

It should be noted that the far-field does not depend on time. The total amount of heat, and hence the total thermal expansion, is constant for $t > 0$. The far-field remains the same as long as the temperature field is concentrated to a limited region around $z = 0$.

The above far-field displacements may be obtained directly from the integrals (46). We let t tend to zero. Then for any $z \neq 0$, $|z'| = |z|/\sqrt{4at} \gg 1$. The upper limit in the integrals (46) tends to infinity: $1/\sqrt{4at} \to \infty$. We have from (46) and (56):

$$\int_0^\infty \frac{\text{erf}(ps)}{s} \cdot \text{e}^{-(r^2-p^2)s^2} ds = \frac{1}{2}\ln\left(\frac{r+p}{r-p}\right) \qquad p = x,\ y$$

$$\sqrt{\pi} z \int_0^\infty \text{erf}(xs) \cdot \text{erf}(ys) \cdot \text{e}^{-z^2 s^2} ds = \arctan\left(\frac{xy}{zr}\right) \qquad (z \neq 0) \tag{57}$$

The first integral is given in Gradstein, Ryzhik (1981, p.650). The second one may be found in Prudnikov et. al. (vol. II, p. 123), which is a huge table of integrals.

The strain and stress fields become for large z':

$$\varepsilon^{qi}_{xx} \simeq \frac{u_0}{r} \cdot \frac{xy}{x^2+z^2} \qquad \varepsilon^{qi}_{yy} \simeq \frac{u_0}{r} \cdot \frac{xy}{y^2+z^2}$$

$$\varepsilon^{qi}_{zz} \simeq -\frac{u_0}{r}\left[\frac{xy}{x^2+z^2} + \frac{xy}{y^2+z^2}\right] \qquad \varepsilon^{qi}_{xy} \simeq -\frac{u_0}{r}$$

$$\varepsilon^{qi}_{xz} \simeq \frac{u_0}{r} \cdot \frac{yz}{x^2+z^2} \qquad \varepsilon^{qi}_{yz} \simeq \frac{u_0}{r} \cdot \frac{xz}{y^2+z^2} \qquad (z' > 2) \tag{58}$$

$$\sigma^{qi}_{xx} \simeq \frac{p_0}{r} \cdot \frac{xy}{x^2+z^2} \qquad \sigma^{qi}_{yy} \simeq \frac{p_0}{r} \cdot \frac{xy}{y^2+z^2}$$

$$\sigma^{qi}_{zz} \simeq -\frac{p_0}{r}\left[\frac{xy}{x^2+z^2} + \frac{xy}{y^2+z^2}\right] \qquad \sigma^{qi}_{xy} \simeq -\frac{p_0}{r}$$

$$\sigma^{qi}_{xz} \simeq \frac{p_0}{r} \cdot \frac{yz}{x^2+z^2} \qquad \sigma^{qi}_{yz} \simeq \frac{p_0}{r} \cdot \frac{xz}{y^2+z^2} \qquad (z' > 2) \tag{59}$$

The strain and stress fields are, except for the scale factors u_0 and p_0, identical, since the temperature T_{qi} and the volume expansion e are put to zero in the far-field approximation.

The above expressions for the strain and stress far-fields may be obtained by direct derivations of the displacement, Eqs.(56) and (18-21). They may also be obtained from the general expressions (50) and (52) as limits for large z'. We have:

$$z' > 2: \qquad r' > 2 \qquad \text{erf}(r') \simeq 1 \qquad \text{e}^{-(z')^2} \simeq 0$$

Insertion in (51) and (8) gives:

$$A(x,r,t) = A'(x',r') \simeq 1 \qquad A(y,r,t) = A(y',r') \simeq 1 \qquad T_{qi}(x,y,z,t) \simeq 0 \tag{60}$$

12 Correction for semi-infinite space

The problem studied this far concerns infinite space $-\infty < z < \infty$. The boundary conditions at the ground surface $z = H$ are not accounted for. The three stress components σ_{zz}, σ_{zx} and σ_{zy} must be zero at the free ground surface, Eq. (5).

The temperature T from the rectangular heat source is superimposed onto the undisturbed ground temperature. The boundary condition for the temperature at the ground surface $z = H$ is that the temperature is zero:

$$T(x,y,H,t) = 0 \tag{61}$$

This boundary condition for the temperature is readily satisfied by the introduction of a mirror heat source with opposite sign at $z = 2H$. We get two quadrantal heat sources, and Eq. (6) is replaced by:

$$\begin{aligned} & e_0 \cdot \mathrm{sign}(x) \cdot \mathrm{sign}(y) \quad \text{over} \quad z = 0 \quad \text{at} \quad t = 0 \\ & -e_0 \cdot \mathrm{sign}(x) \cdot \mathrm{sign}(y) \quad \text{over} \quad z = 2H \quad \text{at} \quad t = 0 \end{aligned} \tag{62}$$

A solution to Navier's equations (2) for the temperature $T = T_{qi}(x, y, z, t) - T_{qi}(x, y, z - 2H, t)$ is obtained by subtracting from the above quadrantal solution the solution with z replaced by $z - 2H$. This solution for the quadrantal heat source and the negative mirror source does not satisfy the boundary conditions (5). The exact values for σ_{zz}, σ_{xz} and σ_{yz} at $z = H$ for the combined solution are given by Eqs. (244)-(246) in Appendix 1. The normal stress at the boundary $z = H$ becomes zero, since the problem with the two quadrantal sources is antisymmetric with respect to the plane $z = H$. The two shear stresses, Eqs. (245)-(246), vary with x and y. They also depend on time t through the factor A'. The time-dependence vanishes when the far-field approximation is used ($A' \simeq 1$, Eq. (60)).

We now make the major assumption that *the far-field approximation may be used at the ground surface.* We have seen in Section 11 that this assumption is valid with good accuracy when the distance H is large compared to the so-called thermal range (which may be defined as $\sqrt{4at}$):

$$H > 2 \cdot \sqrt{4at} \tag{63}$$

The negative mirror heat source gave the remaining boundary-value problem of Appendix 1. The normal stress at the boundary $z = H$ is zero. The two shear stresses are prescribed and equal to the negative values of Eqs. (245)-(246) so that the total solution has zero stresses at $z = H$, Eq. (4). In the far-field approximation, Eq. (63), the boundary values become time-independent.

The remaining problem is to solve Navier's equations (2) (without the temperature term, that is $T = 0$) for the prescribed shear stresses and zero normal stress in a semi-infinite medium. The corresponding problem for prescribed normal stress and zero shear stresses has been solved by Hertz; see Love (1927). The solution involves certain integrals of the prescribed normal stress. See Section 15. In order to be able to use this solution technique, we use a quadrantal mirror heat source of *equal* sign instead of one with opposite sign. We have instead of (62) the following quadrantal heat sources:

$$\begin{aligned} & e_0 \cdot \mathrm{sign}(x) \cdot \mathrm{sign}(y) \quad \text{over} \quad z = 0 \quad \text{at} \quad t = 0 \\ & e_0 \cdot \mathrm{sign}(x) \cdot \mathrm{sign}(y) \quad \text{over} \quad z = 2H \quad \text{at} \quad t = 0 \end{aligned} \tag{64}$$

The solution for the quadrantal mirror heat source is given by the quadrantal solution of the preceding sections with z replaced by $z - 2H$. The two shear stresses for the sum of the original solution and the mirror solution are zero at the ground surface due to symmetry, while the normal stress σ_{zz} is nonzero at $z = H$.

The solution will now consist of three parts:

1. The basic quadrantal solution for infinite space
2. The solution from an equal mirror heat source at $z = 2H$
3. The solution to account for the remaining normal stress distribution at $z = H$

The quadrantal solution for the infinite medium is given above. The mirror solution is given in next section. The remaining problem to determine the response in the semi-infinite space $-\infty < z < H$ for our specific normal stress distribution at $z = H$ is dealt with in Section 14 and onwards. This third part will be called the boundary solution.

The above solution with its three parts is based on the assumption (63). With this assumption, we get a time-independent boundary value problem for the third part. The assumption also means that the temperature field from the heat source at $z = 0$ has not reached the ground surface $z = H$. Likewise, the temperature field from the mirror heat source at $z = 2H$, introduced below and in the next section, has not reached the ground surface $z = H$. This follows from $z' = H/\sqrt{4at} > 2$ and:

$$\mathrm{erf}(2) = 0.995 \simeq 1 \qquad \exp(-2^2) = 0.018 \simeq 0 \tag{65}$$

The assumption (63) implies that the temperature at the ground surface in approximately zero so that the boundary condition for the temperature, Eq. (61), is satisfied.

The solution for the two quadrantal solutions (64) involves another approximation, since the temperature term in Navier's equations is not the exact one. Instead of the exact $T = T_{qi}(x,y,z,t) - T_{qi}(x,y,z-2H,t)$, we use $T = T_{qi}(x,y,z,t) + T_{qi}(x,y,z-2H,t)$. But the temperature $T_{qi}(x,y,z-2H,t)$ from the mirror source is negligible for $z < H$ in the far-field approximation (63).

The main assumption (63) that the far-field approximation can be used, puts restrictions on t, to ensure that the solution is valid:

$$\frac{H}{\sqrt{4at}} > 2 \quad \Leftrightarrow \quad t < \frac{H^2}{16a} \tag{66}$$

The solution is valid in the whole region $(-\infty < z < H)$ when t satisfies this condition. The solution at the repository level is valid during a longer time period since the error occurs slowly at $z = H$. A reasonable criterion for the validity of the solution at the repository level $z = 0$ (and downwards) is:

$$\frac{2H}{\sqrt{4at}} > 2 \quad \Leftrightarrow \quad t < \frac{H^2}{4a} \tag{67}$$

It will take four times longer for the disturbance to reach the repository level than the ground surface. The solution will be totally incorrect for much larger times.

As an example we consider the following case with data from the KBS-3 repository:

$$H = 500 \text{ m} \quad a = 1.62 \cdot 10^{-6} \text{ m}^2/\text{s} \quad \Rightarrow \quad t < \frac{500^2}{16 \cdot 1.62 \cdot 10^{-6}} \text{ s} \simeq 300 \text{ years} \tag{68}$$

The temperature field reaches the ground surface after 300 years for $H = 500$ m. A typical value of the thermal diffusivity a for granite is chosen. The solution derived under this assumption will at least be valid for the first 300 years. At the repository level ($z = 0$) it will roughly take 1200 years or more, before the error due to the far-field approximation becomes noticeable. The solution is not valid for much larger times ($\simeq$10,000).

The three components give the total solution for the instantaneous quadrantal heat source, Eq. (6), in the semi-infinite space $-\infty < z < H$. From this, the solution for the time-dependent heat emission at $z = 0$ is obtained by superposition as an integral in time. Finally, the solution for the rectangular heat source over $-L < x < L$, $-B < y < B$ at $z = 0$ is obtained by superposition in accordance with Eq. (12). The solution with its three parts is taken for x and y replaced by $x \pm L$ and $y \pm B$, respectively.

13 Solution for mirror heat source

We seek the solution for the mirror quadrantal heat source at $z = 2H$:

$$e_0 \cdot \text{sign}(x) \cdot \text{sign}(y) \quad \text{over} \quad z = 2H \quad \text{at} \quad t = 0 \tag{69}$$

The mirror solution is obtained by substituting z with $z - 2H$ in the above solution. The mirror solution is used in the region $z < H$ or $|z - 2H| > H$. Then by assumption (63) we have:

$$\frac{|z-2H|}{\sqrt{4at}} > \frac{H}{\sqrt{4at}} > 2 \tag{70}$$

This means that we may use the far-field expressions of Section 11.

The components of the displacement become from Eq. (56):

$$u^m = -\frac{u_0}{2}\ln\left(\frac{r_m + y}{r_m - y}\right) \qquad v^m = -\frac{u_0}{2}\ln\left(\frac{r_m + x}{r_m - x}\right)$$

$$w^m = u_0 \cdot \arctan\left(\frac{xy}{(z-2H)r_m}\right) \tag{71}$$

The index m is used to denote the solution for the *mirror* quadrantal heat source. The length r_m is the distance to the center of the mirror heat source:

$$r_m = \sqrt{x^2 + y^2 + (z-2H)^2} \tag{72}$$

The strain field is obtained from (58) by replacing r by r_m and z by $z - 2H$:

$$\varepsilon^m_{xx} = \frac{u_0}{r_m}\cdot\frac{xy}{x^2+(z-2H)^2} \qquad \varepsilon^m_{yy} = \frac{u_0}{r_m}\cdot\frac{xy}{y^2+(z-2H)^2}$$

$$\varepsilon^m_{zz} = -\frac{u_0}{r_m}\left[\frac{xy}{x^2+(z-2H)^2} + \frac{xy}{y^2+(z-2H)^2}\right] \qquad \varepsilon^m_{xy} = -\frac{u_0}{r_m}$$

$$\varepsilon^m_{xz} = \frac{u_0}{r_m}\cdot\frac{y(z-2H)}{x^2+(z-2H)^2} \qquad \varepsilon^m_{yz} = \frac{u_0}{r_m}\cdot\frac{x(z-2H)}{y^2+(z-2H)^2} \tag{73}$$

The stress field is obtained by the same substitutions in Eqs. (59). It differs from the strain field by the factor p_0/u_0 only:

$$\sigma^m_{xx} = \frac{p_0}{r_m}\cdot\frac{xy}{x^2+(z-2H)^2} \qquad \sigma^m_{yy} = \frac{p_0}{r_m}\cdot\frac{xy}{y^2+(z-2H)^2}$$

$$\sigma^m_{zz} = -\frac{p_0}{r_m}\left[\frac{xy}{x^2+(z-2H)^2} + \frac{xy}{y^2+(z-2H)^2}\right] \qquad \sigma^m_{xy} = -\frac{p_0}{r_m}$$

$$\sigma^m_{xz} = \frac{p_0}{r_m}\cdot\frac{y(z-2H)}{x^2+(z-2H)^2} \qquad \sigma^m_{yz} = \frac{p_0}{r_m}\cdot\frac{x(z-2H)}{y^2+(z-2H)^2} \tag{74}$$

14 Conditions for boundary solution

The stress field for the quadrantal heat source in an infinite medium is given by Eqs. (52). At the ground surface $z = H$, the far-field expressions (59) may be used. The stress field from the mirror source is given by Eqs. (74). The sum of the two shear stresses at the ground surface are certainly zero:

$$z = H: \qquad \sigma_{xz} = 0 \qquad \sigma_{yz} = 0 \qquad (r = r_m) \tag{75}$$

The normal stress σ_{zz} at $z = H$ from these two parts becomes:

$$\sigma_{zz}(x,y,H,t) = -\frac{2p_0}{r_{0H}}\left(\frac{xy}{x^2+H^2} + \frac{xy}{y^2+H^2}\right) \tag{76}$$

$$r_{0H} = \sqrt{x^2+y^2+H^2} \tag{77}$$

The two contributions for $z = H$ and $z - 2H = -H$ are equal, so there is a factor 2 before p_0.

The third part, the boundary solution, shall balance the above normal stress. We get the boundary condition:

$$\sigma_{zz}^b(x,y,H,t) = \sigma_{b0}\cdot\sigma_b'(x,y,H) \tag{78}$$

The index b refers to the *boundary* solution. Here, we have introduced a dimensionless function σ_b' and a new constant σ_{b0} (Pa) associated with the normal boundary stress:

$$\sigma_{b0} = \frac{2p_0}{H} = \frac{E}{1+\nu}\cdot\frac{2u_0}{H} \tag{79}$$

The normal stress at $z = H$ is determined by the dimensionless normal stress σ_b':

$$\sigma_b'(x,y,H) = \frac{H}{\sqrt{x^2+y^2+H^2}}\left(\frac{xy}{x^2+H^2} + \frac{xy}{y^2+H^2}\right) \tag{80}$$

This dimensionless function depends on x/H and y/H only. It is shown in Fig. 2. The function is odd in x and in y just as T_{qi}, Eq. (8), and it has a maximum in the first (and third) quadrant:

$$\sigma_{b,max}' = \frac{\sqrt{5}-1}{\sqrt{\sqrt{5}+2}} = 0.601 \quad \text{for} \quad x/H = y/H = \sqrt{\frac{\sqrt{5}+1}{2}} = 1.272 \tag{81}$$

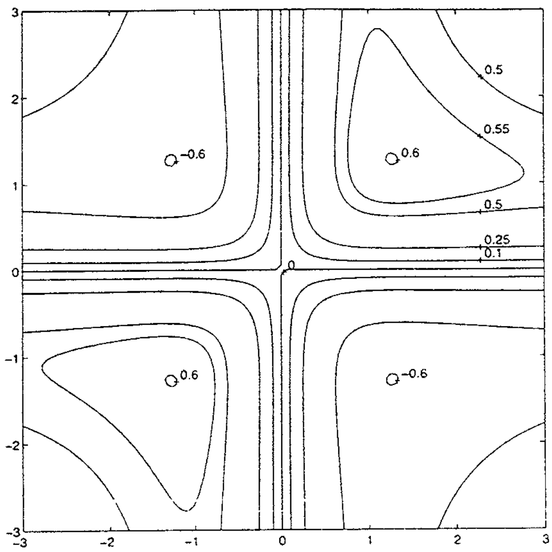

Figure 2. Dimensionless normal stress $\sigma_b'(x,y,H)$ for the boundary solution, Eq. (80).

In Section 6, we used the second-order derivative with respect to x and y. The problem is reduced to that of a point heat source at $x = y = z = 0$. This derivative of $\sigma_b'(x,y,H)$, Eq. (80), becomes:

$$\frac{\partial^2\sigma_b'}{\partial x\partial y} = -\frac{H}{r_{0H}^3} + \frac{3H^3}{r_{0H}^5} \qquad r_{0H} = \sqrt{x^2+y^2+H^2} \tag{82}$$

This relation will be used in Section 16.

Let us summarize the conditions for the boundary solution to correct for the boundary conditions at the ground surface. The displacement $\mathbf{u}_b$ shall satisfy Navier's equation (2) in the *half-space* $-\infty < z < H$ without the temperature terms:

$$\nabla^2(\mathbf{u}_b) + \frac{1}{1-2\nu}\nabla(e_b) = \mathbf{0} \quad e_b = \nabla\cdot\mathbf{u}_b \qquad (-\infty < z < H) \tag{83}$$

The boundary conditions for the boundary correction solution at $z = H$ are:

$$\sigma^b_{xz}(x,y,H) = 0 \quad \sigma^b_{yz}(x,y,H) = 0 \quad \sigma^b_{zz}(x,y,H) = \sigma_{b0}\cdot\sigma'_b(x,y,H) \tag{84}$$

We will solve this semi-infinite problem for the half-space $0 < z < \infty$ and *not* for $-\infty < z < H$. The new displacement $\mathbf{u}_{b0}$ satisfies Navier's equation (2) in the half-space $0 < z < \infty$ without the temperature terms:

$$\nabla^2(\mathbf{u}_{b0}) + \frac{1}{1-2\nu}\nabla(e_{b0}) = \mathbf{0} \quad e_{b0} = \nabla\cdot\mathbf{u}_{b0} \qquad (0 < z < \infty) \tag{85}$$

The boundary conditions for the boundary correction solution at $z = 0$ are:

$$\sigma^{b0}_{xz}(x,y,0) = 0 \quad \sigma^{b0}_{yz}(x,y,0) = 0 \quad \sigma^{b0}_{zz}(x,y,0) = \sigma_{b0}\cdot\sigma'_b(x,y,H) \tag{86}$$

The final boundary solution (index b) is obtained by replacing z by $H - z$ in all formulas for the boundary solution (index $b0$):

$$z \rightarrow H - z \tag{87}$$

The direction of the z-axis is reversed. The displacement w changes its sign:

$$w^b(x,y,z) = -w^{b0}(x,y,H-z) \tag{88}$$

The shear strains ε_{xz} and ε_{yz} and the shear stresses σ_{xz} and σ_{yz} involving the z-coordinate are changed in the same way. We have for example:

$$\sigma^b_{xz}(x,y,z) = -\sigma^{b0}_{xz}(x,y,H-z) \tag{89}$$

15 General formulas for semi-infinite space

General formulas for the solution in the semi-infinite space $0 < z < \infty$, with a prescribed normal stress and zero shear stress at $z = 0$, have been derived by Hertz and others, Love (1927).

The solution is obtained from the following potential due to Hertz:

$$\chi = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dy''\sigma_{zz}(x'',y'',0)\cdot\ln\left(z + \sqrt{(x''-x)^2 + (y''-y)^2 + z^2}\right) \quad z \geq 0 \tag{90}$$

The three components of the displacement are given by:

$$u = \frac{1}{2\mu}\left[z\frac{\partial^2\chi}{\partial x\partial z} + (1-2\nu)\frac{\partial\chi}{\partial x}\right]$$

$$v = \frac{1}{2\mu}\left[z\frac{\partial^2\chi}{\partial y\partial z} + (1-2\nu)\frac{\partial\chi}{\partial y}\right]$$

$$w = \frac{1}{2\mu}\left[z\frac{\partial^2\chi}{\partial z^2} - 2(1-\nu)\frac{\partial\chi}{\partial z}\right] \tag{91}$$

The stress and strain components are then obtained from Eqs. (3) and (4) (putting $T = 0$). The strain components are given by:

$$\varepsilon_{xx} = \frac{1}{2\mu}\left[z\frac{\partial^3\chi}{\partial x^2\partial z} + (1-2\nu)\frac{\partial^2\chi}{\partial x^2}\right] \quad \varepsilon_{yy} = \frac{1}{2\mu}\left[z\frac{\partial^3\chi}{\partial y^2\partial z} + (1-2\nu)\frac{\partial^2\chi}{\partial y^2}\right]$$

$$\varepsilon_{zz} = \frac{1}{2\mu}\left[z\frac{\partial^3\chi}{\partial z^3} - (1-2\nu)\frac{\partial^2\chi}{\partial z^2}\right] \quad \varepsilon_{xy} = \frac{1}{2\mu}\left[z\frac{\partial^3\chi}{\partial x\partial y\partial z} + (1-2\nu)\frac{\partial^2\chi}{\partial x\partial y}\right]$$

$$\varepsilon_{xz} = \frac{1}{2\mu}z\frac{\partial^3\chi}{\partial x\partial z^2} \quad \varepsilon_{yz} = \frac{1}{2\mu}z\frac{\partial^3\chi}{\partial y\partial z^2} \tag{92}$$

The volume expansion is:

$$e = -\frac{1}{2\mu}\cdot 2(1-2\nu)\frac{\partial^2\chi}{\partial z^2} \tag{93}$$

Here, we have used the fact that χ satisfies Laplace equation:

$$\nabla^2\chi = 0 \tag{94}$$

This follows from the fact that $\ln(z + r)$ satisfies Laplace equation. The potential χ is just a superposition of this type of logarithmic solutions, Eq. (90).

The strain components are given by the formulas (4) with $T = 0$:

$$\sigma_{xx} = z\frac{\partial^3\chi}{\partial x^2\partial z} + \frac{\partial^2\chi}{\partial x^2} + 2\nu\frac{\partial^2\chi}{\partial y^2} \quad \sigma_{yy} = z\frac{\partial^3\chi}{\partial y^2\partial z} + \frac{\partial^2\chi}{\partial y^2} + 2\nu\frac{\partial^2\chi}{\partial x^2}$$

$$\sigma_{zz} = z\frac{\partial^3\chi}{\partial z^3} - \frac{\partial^2\chi}{\partial z^2} \quad \sigma_{xy} = z\frac{\partial^3\chi}{\partial x\partial y\partial z} + (1-2\nu)\frac{\partial^2\chi}{\partial x\partial y}$$

$$\sigma_{xz} = z\frac{\partial^3\chi}{\partial x\partial z^2} \quad \sigma_{yz} = z\frac{\partial^3\chi}{\partial y\partial z^2} \tag{95}$$

16 Calculation of Hertz' potential χ

We have to calculate Hertz' potential χ, Eq. (90), with $\sigma_{zz}(x'', y'', 0)$ given by Eq. (78):

$$\chi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dx''\int_{-\infty}^{\infty}dy''\sigma_{b0}\cdot\sigma_b'(x'',y'',H)\cdot\ln(z+\tilde{r})$$

$$\tilde{r} = \sqrt{(x''-x)^2 + (y''-y)^2 + z^2} \tag{96}$$

Here, σ_b' is given by Eq. (80):

$$\sigma_b' = \frac{H}{\sqrt{(x'')^2 + (y'')^2 + H^2}}\left(\frac{x''y''}{(x'')^2 + H^2} + \frac{x''y''}{(y'')^2 + H^2}\right) \tag{97}$$

The double integral is difficult to evaluate.

We first make the variable transformations $\alpha = x'' - x$ and $\beta = y'' - y$. Then we take the derivatives with respect to x and to y. We derive $\sigma_b'(\alpha + x, \beta + y)$. Using Eq. (82) we get:

$$\frac{\partial^2\chi}{\partial x\partial y} = \frac{\sigma_{b0}H}{2\pi}\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\left(\frac{3H^2}{r_{\alpha\beta}^5} - \frac{1}{r_{\alpha\beta}^3}\right)\ln\left(z + \sqrt{\alpha^2+\beta^2+z^2}\right)$$

$$r_{\alpha\beta} = \sqrt{(\alpha + x)^2 + (\beta + y)^2 + H^2} \tag{98}$$

The factor involving $r_{\alpha\beta}$ in the integral may be written as a partial derivative with respect to H (with $\alpha + x$ and $\beta + y$ kept fixed):

$$\frac{\partial}{\partial H}\left(\frac{H}{r_{\alpha\beta}^3}\right) = \frac{1}{r_{\alpha\beta}^3} - \frac{3H^2}{r_{\alpha\beta}^5} \tag{99}$$

So we have:

$$\frac{\partial^2 \chi}{\partial x \partial y} = -\sigma_{b0} H \cdot \frac{\partial}{\partial H}\left[\frac{H}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \frac{\ln\left(z + \sqrt{\alpha^2 + \beta^2 + z^2}\right)}{\sqrt{(\alpha + x)^2 + (\beta + y)^2 + H^2}^3}\right] \tag{100}$$

The above double integral occurs in solutions of Laplace equations for the semi-infinite space ($z > 0$). Consider the following potential problem for $V(x, y, z)$ in $z > 0$:

$$\begin{cases} \nabla^2 V = 0 & z > 0 \\ V(x, y, 0) & \text{prescribed} \end{cases} \tag{101}$$

The general solution is given by the theory of Green functions:

$$V(x, y, z) = \frac{z}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \frac{V(\alpha, \beta, 0)}{\sqrt{(\alpha - x)^2 + (\beta - y)^2 + z^2}^3} \tag{102}$$

This integral is of the type within the brackets in Eq. (100), if z is replaced by H and $V(\alpha, \beta, 0)$ is given by $\ln\left(z_1 + \sqrt{\alpha^2 + \beta^2 + z_1^2}\right)$ with $z_1 = z$.

Now, $\ln(z + r)$ satisfies Laplace equation:

$$\nabla^2[\ln(z + r)] = 0 \quad r = \sqrt{x^2 + y^2 + z^2} \tag{103}$$

We may replace z by $z + z_1$. Let V be given by

$$V(x, y, z) = \ln\left(z + z_1 + \sqrt{x^2 + y^2 + (z + z_1)^2}\right) \tag{104}$$

Then we have:

$$\text{i.} \quad \nabla^2 V = 0 \tag{105}$$

$$\text{ii.} \quad V(x, y, 0) = \ln\left(z_1 + \sqrt{x^2 + y^2 + z_1^2}\right) \tag{106}$$

Thus, we have from formula (102):

$$\ln\left(z + z_1 + \sqrt{x^2 + y^2 + (z + z_1)^2}\right) = \frac{z}{2\pi}\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \frac{\ln\left(z_1 + \sqrt{\alpha^2 + \beta^2 + z_1^2}\right)}{\sqrt{(\alpha - x)^2 + (\beta - y)^2 + z^2}} \tag{107}$$

This double integral is identical to the integral in Eq. (100) after the following substitutions

$$z \to H \quad z_1 \to z \quad x \to -x \quad y \to -y. \tag{108}$$

Thus we have:

$$\frac{\partial^2 \chi}{\partial x \partial y} = -\sigma_{b0} H \cdot \frac{\partial}{\partial H}\left[\ln\left(H + z + \sqrt{(-x)^2 + (-y)^2 + (H + z)^2}\right)\right] \tag{109}$$

After derivation with respect to H, we have the remarkably simple formula:

$$\frac{\partial^2 \chi}{\partial x \partial y} = -\sigma_{b0} H \cdot \frac{1}{\sqrt{x^2 + y^2 + (H+z)^2}} \tag{110}$$

The potential χ is odd in x and in y, and hence zero on the axis $x = 0$ and $y = 0$. This follows from Eq. (90) and the fact that σ_b' is odd in x and in y, Eq. (80). It is obtained from the integral of Eq. (110) from $x = 0$ and $y = 0$:

$$\chi = -\sigma_{b0} H \int_0^x dx'' \int_0^y dy'' \frac{1}{\sqrt{(x'')^2 + (y'')^2 + (H+z)^2}} \tag{111}$$

This double integral has already been evaluated in Section 11. The result becomes, Eqs. (53-54):

$$\chi = -2p_0 \left[\frac{x}{2} \ln\left(\frac{r_H + y}{r_H - y} \right) + \frac{y}{2} \ln\left(\frac{r_H + x}{r_H - x} \right) - (z+H) \arctan\left(\frac{xy}{(z+H) r_H} \right) \right] \tag{112}$$

where

$$r_H = \sqrt{x^2 + y^2 + (z+H)^2} \tag{113}$$

The factor $\sigma_{b0} H$ is equal to $2p_0$, Eq. (79).

17 Derivatives of χ

The boundary solution is obtained from derivatives of χ, Eqs. (91), (92) and (95). The Hertzian potential χ is given by Eq. (112). It depends on the spacial coordinates x, y and z, but it is independent of time t. The distance r_H is defined by Eq. (113).

The first-order derivatives of $\chi(x, y, z)$, Eq. (112), become in the same way as in Eqs.(55-56):

$$\frac{\partial \chi}{\partial x} = -p_0 \cdot \ln\left(\frac{r_H + y}{r_H - y} \right) \qquad \frac{\partial \chi}{\partial y} = -p_0 \cdot \ln\left(\frac{r_H + x}{r_H - x} \right) \tag{114}$$

$$\frac{\partial \chi}{\partial z} = 2p_0 \cdot \arctan\left(\frac{xy}{(z+H) r_H} \right) \tag{115}$$

The second-order derivatives of $\chi(x, y, z)$ become:

$$\frac{\partial^2 \chi}{\partial x^2} = 2p_0 \cdot \frac{1}{r_H} \cdot \frac{xy}{x^2 + (z+H)^2} \qquad \frac{\partial^2 \chi}{\partial y^2} = 2p_0 \cdot \frac{1}{r_H} \cdot \frac{xy}{y^2 + (z+H)^2} \tag{116}$$

$$\frac{\partial^2 \chi}{\partial z^2} = -2p_0 \cdot \frac{1}{r_H} \left(\frac{xy}{x^2 + (z+H)^2} + \frac{xy}{y^2 + (z+H)^2} \right) \tag{117}$$

$$\frac{\partial^2 \chi}{\partial x \partial y} = -2p_0 \cdot \frac{1}{r_H} \tag{118}$$

$$\frac{\partial^2 \chi}{\partial x \partial z} = 2p_0 \cdot \frac{1}{r_H} \cdot \frac{y(z+H)}{x^2 + (z+H)^2} \tag{119}$$

$$\frac{\partial^2 \chi}{\partial y \partial z} = 2p_0 \cdot \frac{1}{r_H} \cdot \frac{x(z+H)}{y^2 + (z+H)^2} \tag{120}$$

The sum of the three first second-order derivatives is zero, which verifies that χ satisfies Laplace equation: $\nabla^2 \chi = 0$.

We need the following third-order derivatives of $\chi(x, y, z)$:

$$\frac{\partial^3 \chi}{\partial x \partial y \partial z} = 2p_0 \cdot \frac{z+H}{r_H^3} \tag{121}$$

$$\frac{\partial^3 \chi}{\partial x^2 \partial z} = -2p_0 \cdot \frac{z+H}{r_H} \cdot \frac{xy}{x^2+(z+H)^2}\left(\frac{1}{r_H^2}+\frac{2}{x^2+(z+H)^2}\right) \tag{122}$$

$$\frac{\partial^3 \chi}{\partial y^2 \partial z} = -2p_0 \cdot \frac{z+H}{r_H} \cdot \frac{xy}{y^2+(z+H)^2}\left(\frac{1}{r_H^2}+\frac{2}{y^2+(z+H)^2}\right) \tag{123}$$

$$\frac{\partial^3 \chi}{\partial x \partial z^2} = 2p_0 \cdot \frac{y}{r_H} \cdot \frac{1}{x^2+(z+H)^2}\left(1-(z+H)^2\left(\frac{1}{r_H^2}+\frac{2}{x^2+(z+H)^2}\right)\right) \tag{124}$$

$$\frac{\partial^3 \chi}{\partial y \partial z^2} = 2p_0 \cdot \frac{x}{r_H} \cdot \frac{1}{y^2+(z+H)^2}\left(1-(z+H)^2\left(\frac{1}{r_H^2}+\frac{2}{y^2+(z+H)^2}\right)\right) \tag{125}$$

$$\frac{\partial^3 \chi}{\partial z^3} = 2p_0 \cdot \frac{z+H}{r_H}\left[\frac{xy}{x^2+(z+H)^2}\left(\frac{1}{r_H^2}+\frac{2}{x^2+(z+H)^2}\right)+\right.$$
$$\left.+\frac{xy}{y^2+(z+H)^2}\left(\frac{1}{r_H^2}+\frac{2}{y^2+(z+H)^2}\right)\right] \tag{126}$$

It is easy to verify that $\frac{\partial}{\partial z}\nabla^2\chi = 0$.

18 Boundary solution

The boundary solution in the region $z > 0$ is obtained by insertion of the derivatives of χ in Eqs. (91), (92) and (95). The obtained expressions for all displacement, strain and stress components are given in Appendix 2. This boundary solution for the region $z > 0$ has the index $b0$.

The final boundary solution for the region $z < H$ is obtained by the substitution (87), where z is replaced by $H-z$. The changes (88) and (89) must also be performed. We have for example:

$$u^b(x,y,z) = u^{b0}(x,y,H-z) \qquad w^b(x,y,z) = -w^{b0}(x,y,H-z) \tag{127}$$

The radius r_H, Eq. (113), becomes equal to r_m, Eq. (72):

$$r_H|_{z\to H-z} = \sqrt{x^2+y^2+(H-z+H)^2} = \sqrt{x^2+y^2+(2H-z)^2} = r_m \tag{128}$$

To simplify the solution two commonly occurring denominators will be replaced by D_x and D_y (m^2). The two denominators are:

$$D_x = x^2+(2H-z)^2 \qquad D_y = y^2+(2H-z)^2 \tag{129}$$

The three displacement components become from Eqs. (247-249):

$$u^b = 2u_0\left[\frac{H-z}{r_m}\cdot\frac{y(2H-z)}{D_x} - \frac{1-2\nu}{2}\ln\left(\frac{r_m+y}{r_m-y}\right)\right] \tag{130}$$

$$v^b = 2u_0\left[\frac{H-z}{r_m}\cdot\frac{x(2H-z)}{D_y} - \frac{1-2\nu}{2}\ln\left(\frac{r_m+x}{r_m-x}\right)\right] \tag{131}$$

$$w^b = 2u_0\left[\frac{H-z}{r_m}\left(\frac{xy}{D_x}+\frac{xy}{D_y}\right)+2(1-\nu)\arctan\left(\frac{xy}{(2H-z)r_m}\right)\right] \tag{132}$$

The strain field becomes from Eqs. (250-255):

$$\varepsilon_{xx}^b = 2u_0\cdot\frac{1}{r_m}\cdot\frac{xy}{D_x}\left[1-2\nu-(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_x}\right)\right] \tag{133}$$

$$\varepsilon_{yy}^b = 2u_0\cdot\frac{1}{r_m}\cdot\frac{xy}{D_y}\left[1-2\nu-(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_y}\right)\right] \tag{134}$$

$$\varepsilon_{zz}^b = 2u_0\cdot\frac{1}{r_m}\left[\frac{xy}{D_x}\left(1-2\nu+(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_x}\right)\right)+\right.$$

$$\left.+\frac{xy}{D_y}\left(1-2\nu+(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_y}\right)\right)\right] \tag{135}$$

$$\varepsilon_{xy}^b = 2u_0\cdot\frac{1}{r_m}\left[\frac{(H-z)(2H-z)}{r_m^2}-1+2\nu\right] \tag{136}$$

$$\varepsilon_{xz}^b = -2u_0\cdot\frac{1}{r_m}\cdot\frac{y(H-z)}{D_x}\left(1-(2H-z)^2\left(\frac{1}{r_m^2}+\frac{2}{D_x}\right)\right) \tag{137}$$

$$\varepsilon_{yz}^b = -2u_0\cdot\frac{1}{r_m}\cdot\frac{x(H-z)}{D_y}\left(1-(2H-z)^2\left(\frac{1}{r_m^2}+\frac{2}{D_y}\right)\right) \tag{138}$$

The stress field becomes from Eqs. (256-259):

$$\sigma_{xx}^b = 2p_0\cdot\frac{1}{r_m}\left[\frac{xy}{D_x}\left(1-(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_x}\right)\right)+2\nu\frac{xy}{D_y}\right] \tag{139}$$

$$\sigma_{yy}^b = 2p_0\cdot\frac{1}{r_m}\left[\frac{xy}{D_y}\left(1-(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_y}\right)\right)+2\nu\frac{xy}{D_x}\right] \tag{140}$$

$$\sigma_{zz}^b = 2p_0\cdot\frac{1}{r_m}\left[\frac{xy}{D_x}\left(1+(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_x}\right)\right)+\right.$$

$$\left.+\frac{xy}{D_y}\left(1+(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_y}\right)\right)\right] \tag{141}$$

$$\sigma_{xy}^b = \frac{p_0}{u_0}\cdot\varepsilon_{xy}^b \qquad \sigma_{xz}^b = \frac{p_0}{u_0}\cdot\varepsilon_{xz}^b \qquad \sigma_{yz}^b = \frac{p_0}{u_0}\cdot\varepsilon_{yz}^b \tag{142}$$

Insertion of $z = H$ into the expressions for σ_{xz}^b, σ_{yz}^b, and σ_{zz}^b verifies that the boundary conditions (84) are indeed fulfilled, since $2p_0 = \sigma_{b0}\cdot H$, Eq. (79).

19 Quadrantal solution for semi-infinite space

The total solution for the *quadrantal*, instantaneous heat source (6) in the *semi-infinite* space $-\infty < z < H$ (index qs), is obtained from the solution of the infinite problem $-\infty < z < \infty$ (index qi), the mirror solution with the quadrantal heat source at the plane $z = 2H$ (index m), and the boundary solution to correct for the boundary conditions at $z = H$ (index b). We have for example:

$$u^{qs}(x,y,z,t) = u^{qi}(x,y,z,t) + u^{mb}(x,y,z) \tag{143}$$

Here, the index mb stands for the sum of the *mirror* and *boundary* solutions:

$$u^{mb}(x,y,z) = u^{m}(x,y,z) + u^{b}(x,y,z) \tag{144}$$

All components of the displacement, strain, and stress fields are obtained by formulas of the above type. We have for any component f of these three fields:

$$f^{qs}(x,y,z,t) = f^{qi}(x,y,z,t) + f^{mb}(x,y,z)$$

$$f^{mb}(x,y,z) = f^{m}(x,y,z) + f^{b}(x,y,z)$$

$$f = u,\ v,\ w,\ \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz},\ \varepsilon_{yz},\ \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz},\ \text{or } \sigma_{yz} \tag{145}$$

The total solution qs is in this way separated into a time-dependent and time-independent part, with the index qi and mb, respectively. This separation will simplify time integration in Section 20.

The expressions for the three components of the solution (with index qi, m, and b) contain some secondary quantities. The quantities defined in Eqs. (51), (8), (72), and (129) will be used:

$$A(p,r,t) = \operatorname{erf}\left(\frac{r}{\sqrt{4at}}\right) - r\cdot e^{-(r^2-p^2)/(4at)}\cdot\frac{\operatorname{erf}(p/\sqrt{4at})}{p} \qquad p = x,\ y$$

$$T_{qi}(x,y,z,t) = \frac{e_0}{\rho c}\cdot\frac{1}{\sqrt{4\pi at}}\cdot\operatorname{erf}\left(\frac{x}{\sqrt{4at}}\right)\cdot\operatorname{erf}\left(\frac{y}{\sqrt{4at}}\right)\cdot\exp\left(-\frac{z^2}{4at}\right)$$

$$u_0 = \frac{1+\nu}{1-\nu}\cdot\frac{e_0\alpha}{\pi\rho c} \qquad p_0 = \frac{E}{1+\nu}\cdot u_0 = \frac{E}{1-\nu}\cdot\frac{e_0\alpha}{\pi\rho c}$$

$$r_m = \sqrt{x^2+y^2+(2H-z)^2} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$D_x = x^2+(2H-z)^2 \qquad D_y = y^2+(2H-z)^2 \tag{146}$$

We will also use the new quantities B_x and B_y given by:

$$B_x = 2(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_x}\right) \tag{147}$$

$$B_y = 2(H-z)(2H-z)\left(\frac{1}{r_m^2}+\frac{2}{D_y}\right) \tag{148}$$

The **displacement** is given by Eqs. (46), (71) and (130)-(132). The time-dependent part of the displacements is, Eqs. (46):

$$u^{qi}(x,y,z,t) = -u_0\int_0^{1/\sqrt{4at}}\frac{\operatorname{erf}(ys)}{s}\cdot e^{-(r^2-y^2)s^2}\,ds$$

$$v^{qi}(x,y,z,t) = -u_0\int_0^{1/\sqrt{4at}}\frac{\operatorname{erf}(xs)}{s}\cdot e^{-(r^2-x^2)s^2}\,ds$$

$$w^{qi}(x,y,z,t) = u_0\sqrt{\pi}z\int_0^{1/\sqrt{4at}}\operatorname{erf}(xs)\cdot\operatorname{erf}(ys)\cdot e^{-z^2s^2}\,ds \tag{149}$$

The time-independent part of the displacements is obtained from the mirror and boundary solutions (index mb). We have from Eqs. (71) and (130-132):

$$u^{mb}(x,y,z) = u_0\left[-\left(\frac{3}{2}-2\nu\right)\ln\left(\frac{r_m+y}{r_m-y}\right)+\frac{2(H-z)}{r_m}\cdot\frac{y(2H-z)}{D_x}\right]$$

$$v^{mb}(x,y,z) = u_0\left[-\left(\frac{3}{2}-2\nu\right)\ln\left(\frac{r_m+x}{r_m-x}\right)+\frac{2(H-z)}{r_m}\cdot\frac{x(2H-z)}{D_y}\right]$$

$$w^{mb}(x,y,z) = u_0\left[(3-4\nu)\arctan\left(\frac{xy}{(2H-z)r_m}\right)+\frac{2(H-z)}{r_m}\left(\frac{xy}{D_x}+\frac{xy}{D_y}\right)\right] \quad (150)$$

The **strain field** is given by Eqs. (50), (73) and (133-138). The time-dependent part of the displacements is, Eqs. (50):

$$\varepsilon_{xx}^{qi} = \frac{u_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot A(y,r,t) \qquad \varepsilon_{yy}^{qi} = \frac{u_0}{r}\cdot\frac{xy}{y^2+z^2}\cdot A(x,r,t)$$

$$\varepsilon_{zz}^{qi} = \frac{1+\nu}{1-\nu}\alpha\cdot T_{qi}(x,y,z,t)-\frac{u_0}{r}\left[\frac{xy}{x^2+z^2}\cdot A(y,r,t)+\frac{xy}{y^2+z^2}\cdot A(x,r,t)\right]$$

$$\varepsilon_{xz}^{qi} = \frac{u_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot A(y,r,t) \qquad \varepsilon_{yz}^{qi} = \frac{u_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot A(x,r,t)$$

$$\varepsilon_{xy}^{qi} = -\frac{u_0}{r}\cdot \operatorname{erf}\left(\frac{r}{\sqrt{4at}}\right) \quad (151)$$

The time-independent components of the strain field are from Eqs. (73) and (133-138):

$$\varepsilon_{xx}^{mb} = \frac{u_0}{r_m}\cdot\frac{xy}{D_x}(3-4\nu-B_x)$$

$$\varepsilon_{yy}^{mb} = \frac{u_0}{r_m}\cdot\frac{xy}{D_y}(3-4\nu-B_y)$$

$$\varepsilon_{zz}^{mb} = \frac{u_0}{r_m}\left[\frac{xy}{D_x}(1-4\nu+B_x)+\frac{xy}{D_y}(1-4\nu+B_y)\right]$$

$$\varepsilon_{xy}^{mb} = \frac{u_0}{r_m}\left[\frac{2(H-z)(2H-z)}{r_m^2}-3+4\nu\right]$$

$$\varepsilon_{xz}^{mb} = -\frac{u_0}{r_m}\left[\frac{y(4H-3z)}{D_x}-\frac{y(2H-z)}{D_x}\cdot B_x\right]$$

$$\varepsilon_{yz}^{mb} = -\frac{u_0}{r_m}\left[\frac{x(4H-3z)}{D_y}-\frac{x(2H-z)}{D_y}\cdot B_y\right] \quad (152)$$

The **stress field** is given by Eqs. (52), (74) and (139-142). The time-dependent part is given by Eqs. (52):

$$\sigma_{xx}^{qi} = \frac{p_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot A(y,r,t)-\frac{E\alpha}{1-\nu}\cdot T_{qi}(x,y,z,t)$$

$$\sigma_{yy}^{qi} = \frac{p_0}{r}\cdot\frac{xy}{y^2+z^2}\cdot A(x,r,t)-\frac{E\alpha}{1-\nu}\cdot T_{qi}(x,y,z,t)$$

$$\sigma_{zz}^{qi} = -\frac{p_0}{r}\left[\frac{xy}{x^2+z^2}\cdot A(y,r,t) + \frac{xy}{y^2+z^2}\cdot A(x,r,t)\right]$$

$$\sigma_{xz}^{qi} = \frac{p_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot A(y,r,t) \qquad \sigma_{yz}^{qi} = \frac{p_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot A(x,r,t)$$

$$\sigma_{xy}^{qi} = -\frac{p_0}{r}\cdot \mathrm{erf}\left(\frac{r}{\sqrt{4at}}\right) \tag{153}$$

The time-independent components of the stress field are from Eqs. (74) and (139-142):

$$\sigma_{xx}^{mb} = \frac{p_0}{r_m}\left[\frac{xy}{D_x}(3-B_x) + 4\nu\frac{xy}{D_y}\right]$$

$$\sigma_{yy}^{mb} = \frac{p_0}{r_m}\left[\frac{xy}{D_y}(3-B_y) + 4\nu\frac{xy}{D_x}\right]$$

$$\sigma_{zz}^{mb} = \frac{p_0}{r_m}\left[\frac{xy}{D_x}(1+B_x) + \frac{xy}{D_y}(1+B_y)\right]$$

$$\sigma_{xy}^{mb} = \frac{p_0}{r_m}\left[\frac{2(H-z)(2H-z)}{r_m^2} - 3 + 4\nu\right]$$

$$\sigma_{xz}^{mb} = -\frac{p_0}{r_m}\left[\frac{y(4H-3z)}{D_x} - \frac{y(2H-z)}{D_x}\cdot B_x\right]$$

$$\sigma_{yz}^{mb} = -\frac{p_0}{r_m}\cdot\left[\frac{x(4H-3z)}{D_y} - \frac{x(2H-z)}{D_y}\cdot B_y\right] \tag{154}$$

The shear strains and stresses differ only by the factors u_0 and p_0.

20 Time-dependent heat source

Until now the heat emission e_0 (J/m^2) has been instantaneous (at $t = 0$). Although all the components of the displacement, strain, and stress fields of the total quadrantal solution have been derived from a temperature field caused by an instantaneous heat emission, it is fairly easy to change from this instantaneous case to any time-dependent heat emission.

20.1 General formulas for quadrantal solution

Let $q(t)$ (W/m^2) be any time-dependent heat source. We want the solution at the time t. The heat emission during a time increment dt' is $q(t')\cdot dt'$. The response to this instantaneous heat source is given by our solution taken at the time $t - t'$. We have to multiply the solution by $q(t')dt'/e_0$, since the instantaneous solution has the heat emission e_0. The total solution at t is obtained by integration over the interval $0 < t' < t$.

Consider as an example the stress component σ_{xx}. We have for any time-dependent heat source $q(t)$:

$$\sigma_{xx}^{qq}(x,y,z,t) = \int_0^t \frac{q(t')}{e_0}\cdot\sigma_{xx}^{qs}(x,y,z,t-t')\mathrm{d}t' \tag{155}$$

Here, the upper index qq refers to *quadrantal* solution for a *heat source* $q(t)$, while index qs refers to the quadrantal solution for the instantaneous heat source in the semi-infinite space.

The above type of superposition is valid for all components of the displacement, strain and stress fields. It is often called Duhamel's theorem, Carslaw and Jaeger (1959). The function $q(t)/e_0$ enters into the Duhamel integral. We introduce a new notation for this function:

$$q(t) = e_0 \cdot \tilde{q}(t) \tag{156}$$

The constant e_0 (J/m^2) cancels in the Duhamel integrals of the type (155) since the solution (with index qs) is proportional to e_0. The value of e_0 becomes redundant, and it may be chosen at will ($e_0 \neq 0$). The time-dependent part $\tilde{q}(t)$ has the dimension 1/s.

Let f denote any component of the displacement, strain field or stress field. The Duhamel integral involves $f^{qs} = f^{qi} + f^{mb}$, Eq. (145). The second part is independent of time. We have the general formula:

$$f^{qq}(x,y,z,t) = \int_0^t \tilde{q}(t') \cdot f^{qi}(x,y,z,t-t')\,\mathrm{d}t + I_q(t) \cdot f^{mb}(x,y,z)$$

$$f = u,\ v,\ w,\ \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz},\ \varepsilon_{yz},\ \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz},\ \text{or}\ \sigma_{yz} \tag{157}$$

The expressions for f^{qi} and f^{mb} are given by (149-154). The integral of $\tilde{q}(t')$ is introduced:

$$I_q(t) = \int_0^t \tilde{q}(t')\mathrm{d}t' = \frac{1}{e_0}\int_0^t q(t')\mathrm{d}t' \tag{158}$$

20.2 Time integrals

The time t occurs in the factor $I_q(t)$ of the second part in Eq. (157) and in the first integral. The displacements are treated below. For the components of the strain and stress fields, we get integrals of $\tilde{q}(t')$ times the time-dependent factors in the formulas (151) and (153) for strain and stress. The following integrals are needed:

$$I_{qe}(r,t) = \int_0^t \tilde{q}(t') \cdot \mathrm{erf}\left(\frac{r}{\sqrt{4a(t-t')}}\right)\mathrm{d}t' \tag{159}$$

$$I_{qA}(x,r,t) = \int_0^t \tilde{q}(t') \cdot A\,(x,r,t-t')\,\mathrm{d}t' \tag{160}$$

$$I_{qA}(y,r,t) = \int_0^t \tilde{q}(t') \cdot A\,(y,r,t-t')\,\mathrm{d}t' \tag{161}$$

$$I_{qT}(x,y,z,t) = \int_0^t \tilde{q}(t') \cdot \frac{1+\nu}{1-\nu}\alpha \cdot T_{qi}(x,y,z,t-t')\mathrm{d}t' \tag{162}$$

The integrals $I_{qA}(x,r,t)$ and $I_{qA}(y,r,t)$ differ only by the first argument. All the integrals are dimensionless.

The integral involving A, (146), may be written:

$$I_{qA}(p,r,t) = I_{qe}(r,t) - \frac{r}{p}\int_0^t \tilde{q}(t') \cdot \exp\left(-\frac{r^2-p^2}{4a(t-t')}\right) \cdot \mathrm{erf}\left(\frac{p}{\sqrt{4a(t-t')}}\right)\mathrm{d}t' \quad p = x,\ y \tag{163}$$

The integral involving T_{qi}, (146), may be written:

$$I_{qT}(x,y,z,t) = \int_0^t \frac{\tilde{q}(t') \cdot u_0\sqrt{\pi}}{\sqrt{4a(t-t')}}\,\mathrm{erf}\left(\frac{x}{\sqrt{4a(t-t')}}\right)\mathrm{erf}\left(\frac{y}{\sqrt{4a(t-t')}}\right)\exp\left(\frac{-z^2}{4a(t-t')}\right)\mathrm{d}t' \tag{164}$$

The above integrals are simplified somewhat with the following substitution:

$$s = \frac{1}{\sqrt{4a(t-t')}} \qquad t - t' = \frac{1}{4as^2} \qquad dt' = \frac{ds}{2as^3} \tag{165}$$

Then we get:

$$I_{qe}(r,t) = \int_{1/\sqrt{4at}}^{\infty} \tilde{q}[t - 1/(4as^2)] \cdot \frac{\mathrm{erf}(rs)}{2as^3} \mathrm{d}s \tag{166}$$

$$I_{qA}(p,r,t) = I_{qe}(r,t) - \frac{r}{p} \int_{1/\sqrt{4at}}^{\infty} \tilde{q}[t - 1/(4as^2)] \cdot \mathrm{e}^{-(r^2-p^2)s^2} \cdot \frac{\mathrm{erf}(ps)}{2as^3} \mathrm{d}s \quad p = x,\, y \tag{167}$$

$$I_{qT}(x,y,z,t) = \int_{1/\sqrt{4at}}^{\infty} \tilde{q}[t - 1/(4as^2)] \cdot \frac{u_0\sqrt{\pi}}{2as^2} \cdot \mathrm{erf}(xs) \cdot \mathrm{erf}(ys) \cdot \mathrm{e}^{-z^2s^2} \mathrm{d}s \tag{168}$$

20.3 Displacement

The three displacement components are obtained by insertion of Eqs. (149) and (150) in Eq. (157). For the u-component we have:

$$u^{qq} = -u_0 \int_0^t \tilde{q}(t') \left(\int_0^{1/\sqrt{4a(t-t')}} \frac{\mathrm{erf}(ys)}{s} \cdot \mathrm{e}^{-(r^2-y^2)s^2} ds \right) \mathrm{d}t' + I_q(t) \cdot u^{mb}(x,y,z) \tag{169}$$

The double integral may be transformed into a single integral by partial integration in t'. The integral of $\tilde{q}$ is I_q, Eq. (158). We have:

$$\int_0^t \tilde{q}(t') \left(\int_0^{1/\sqrt{4a(t-t')}} \frac{\mathrm{erf}(ys)}{s} \cdot \mathrm{e}^{-(r^2-y^2)s^2} ds \right) \mathrm{d}t' =$$

$$\left[I_q(t') \left(\int_0^{1/\sqrt{4a(t-t')}} \frac{\mathrm{erf}(ys)}{s} \cdot \mathrm{e}^{-(r^2-y^2)s^2} ds \right) \mathrm{d}t' \right]_{t'=0}^{t'=t} -$$

$$\int_0^t I_q(t') \cdot \mathrm{erf}\left(\frac{y}{\sqrt{4a(t-t')}} \right) \cdot \exp\left(-\frac{r^2-y^2}{4a(t-t')} \right) \frac{1}{2(t-t')} \mathrm{d}t' \tag{170}$$

Insertion of $t' = t$ in the first part gives the first infinite integral in Eq. (57). In the second integral we use the substitution (165). The v-component is obtained in the same way (x and y change place). The double integral for w is in the same way reduced to a single integral by a partial integration in t'. Here the second infinite integral in (57) is used. We get the following general formulas for the displacement:

$$u^{qq}(x,y,z,t) = I_q(t) \cdot \left[-\frac{u_0}{2} \ln\left(\frac{r+y}{r-y} \right) + u^{mb}(x,y,z) \right] +$$

$$+ u_0 \int_{1/\sqrt{4at}}^{\infty} I_q[t - 1/(4as^2)] \cdot \frac{\mathrm{erf}(ys)}{s} \cdot \mathrm{e}^{-(r^2-y^2)s^2} ds \tag{171}$$

$$v^{qq}(x,y,z,t) = I_q(t) \cdot \left[-\frac{u_0}{2} \ln\left(\frac{r+x}{r-x} \right) + v^{mb}(x,y,z) \right] +$$

$$+ u_0 \int_{1/\sqrt{4at}}^{\infty} I_q[t - 1/(4as^2)] \cdot \frac{\mathrm{erf}(xs)}{s} \cdot \mathrm{e}^{-(r^2-x^2)s^2} ds \tag{172}$$

$$w^{qq}(x,y,z,t) = I_q(t) \cdot \left[u_0 \arctan\left(\frac{xy}{zr} \right) + w^{mb}(x,y,z) \right] -$$

$$u_0 z\sqrt{\pi}\int_{1/\sqrt{4at}}^{\infty} I_q[t - 1/(4as^2)]\cdot \mathrm{erf}(xs)\mathrm{erf}(ys)\cdot e^{-z^2s^2}\,ds \tag{173}$$

Here, $I_q(t)$ is defined by Eq. (158), u_0 by Eq. (146), and u^{mb}, v^{mb} and w^{mb} by Eqs. (150).

20.4 Strain field

The six components of the strain field are obtained by insertion of Eqs. (151) and (152) in Eq. (157). The time integrals are given in Subsection 20.2. We have:

$$\varepsilon_{xx}^{qq} = \frac{u_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot I_{qA}(y,r,t) + I_q(t)\cdot\varepsilon_{xx}^{mb}(x,y,z)$$

$$\varepsilon_{yy}^{qq} = \frac{u_0}{r}\cdot\frac{xy}{y^2+z^2}\cdot I_{qA}(x,r,t) + I_q(t)\cdot\varepsilon_{yy}^{mb}(x,y,z)$$

$$\varepsilon_{zz}^{qq} = -\frac{u_0}{r}\left[\frac{xy}{x^2+z^2}\cdot I_{qA}(y,r,t) + \frac{xy}{y^2+z^2}\cdot I_{qA}(x,r,t)\right] +$$

$$+ I_{qT}(x,y,z,t) + I_q(t)\cdot\varepsilon_{zz}^{mb}(x,y,z)$$

$$\varepsilon_{xz}^{qq} = \frac{u_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot I_{qA}(y,r,t) + I_q(t)\cdot\varepsilon_{xz}^{mb}(x,y,z)$$

$$\varepsilon_{yz}^{qq} = \frac{u_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot I_{qA}(x,r,t) + I_q(t)\cdot\varepsilon_{yz}^{mb}(x,y,z)$$

$$\varepsilon_{xy}^{qq} = -\frac{u_0}{r}\cdot I_{qe}(r,t) + I_q(t)\cdot\varepsilon_{xy}^{mb}(x,y,z) \tag{174}$$

Here, $I_{qA}(y,r,t)$ is given by Eq. (163), $I_{qT}(x,y,z,t)$ by Eq. (164), $I_q(t)$ by Eq. (158), I_{qe} by (159) and $\varepsilon_{..}^{mb}(x,y,z)$ by Eqs. (152).

20.5 Stress field

The six components of the stress field are obtained by insertion of Eqs. (153) and (154) in Eq. (157).

$$\sigma_{xx}^{qq} = \frac{p_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot I_{qA}(y,r,t) - \frac{E}{1+\nu}\cdot I_{qT}(x,y,z,t) + I_q(t)\cdot\sigma_{xx}^{mb}(x,y,z)$$

$$\sigma_{yy}^{qq} = \frac{p_0}{r}\cdot\frac{xy}{y^2+z^2}\cdot I_{qA}(x,r,t) - \frac{E}{1+\nu}\cdot I_{qT}(x,y,z,t) + I_q(t)\cdot\sigma_{yy}^{mb}(x,y,z)$$

$$\sigma_{zz}^{qq} = -\frac{p_0}{r}\left[\frac{xy}{x^2+z^2}\cdot I_{qA}(y,r,t) + \frac{xy}{y^2+z^2}\cdot I_{qA}(x,r,t)\right] + I_q(t)\cdot\sigma_{zz}^{mb}(x,y,z)$$

$$\sigma_{xz}^{qq} = \frac{p_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot I_{qA}(y,r,t) + I_q(t)\cdot\sigma_{xz}^{mb}(x,y,z)$$

$$\sigma_{yz}^{qq} = \frac{p_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot I_{qA}(x,r,t) + I_q(t)\cdot\sigma_{yz}^{mb}(x,y,z)$$

$$\sigma_{xy}^{qq} = -\frac{p_0}{r}\cdot I_{qe}(r,t) + I_q(t)\cdot\sigma_{xy}^{mb}(x,y,z) \tag{175}$$

Here, $I_{qA}(y,r,t)$ is given by Eq. (163), $I_{qT}(x,y,z,t)$ by Eq. (164), $I_q(t)$ by Eq. (158), I_{qe} by (159) and $\sigma_{..}^{mb}(x,y,z)$ by Eqs. (154).

20.6 General formulas for rectangular heat source

The solution for any time-dependent heat source $q(t)$ (W/m^2) in the rectangular area $-L < x < L$, $-B < y < B$, $z = 0$ in the semi-infinite space $-\infty < z < H$ is obtained by superposition of four quadrantal solutions as described in Section 3. We have in accordance with Eq. (13):

$$f(x,y,z,t) = \sum_{n_x=\pm 1} \sum_{n_y=\pm 1} \frac{n_x n_y}{4} \cdot f^{qq}(x + n_x L, y + n_y B, z, t)$$

$$f = u,\ v,\ w,\ \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz},\ \varepsilon_{yz},\ \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz},\ \text{or}\ \sigma_{yz} \tag{176}$$

The summation indices n_x and n_y assume the values $+1$ and -1 only.

21 Exponentially decaying heat source

Let us now consider the case when $q(t)$ consists of a single exponential:

$$q(t) = q_1 \cdot \mathrm{e}^{-t/t_1} \qquad \mathrm{W/m^2} \tag{177}$$

The amount of heat emission until time t is:

$$\int_0^t q(t')\mathrm{d}t' = q_1 t_1 \cdot (1 - \mathrm{e}^{-t/t_1}) \qquad \mathrm{J/m^2} \tag{178}$$

The total amount of heat emitted until $t = \infty$ is $q_1 t_1$ ($t_1 > 0$).

21.1 Time integrals

The heat source function $\tilde{q}(t)$ is:

$$\tilde{q}(t) = \frac{q(t)}{e_0} = \frac{q_1}{e_0} \cdot \mathrm{e}^{-t/t_1} = \frac{q_1 t_1}{e_0} \cdot \frac{1}{t_1} \cdot \mathrm{e}^{-t/t_1} \qquad (1/\mathrm{s}) \tag{179}$$

The integral of $\tilde{q}(t')$ becomes:

$$I_e(t;1) = \int_0^t \tilde{q}(t')\mathrm{d}t' = \frac{q_1 t_1}{e_0} \cdot (1 - \mathrm{e}^{-t/t_1}) \tag{180}$$

The integrals (159), (163) and (164) become:

$$I_{ee}(r,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^t \frac{1}{t_1} \mathrm{e}^{-t'/t_1} \cdot \mathrm{erf}\left(\frac{r}{\sqrt{4a(t-t')}}\right) \mathrm{d}t' \tag{181}$$

$$I_{eA}(p,r,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^t \frac{1}{t_1} \mathrm{e}^{-t'/t_1} \cdot A(p,r,t-t')\mathrm{d}t' \qquad p = x,\ y \tag{182}$$

$$I_{eT}(x,y,z,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^t \frac{1}{t_1} \mathrm{e}^{-t'/t_1} \cdot \frac{1+\nu}{1-\nu} \alpha \cdot T_{qi}(x,y,z,t-t')\mathrm{d}t' \tag{183}$$

The parameter 1 in the argument list of the four integrals above denotes that the exponential (177) with the decay time t_1 and the amplitude q_1 is involved.

The above integrals get a somewhat simpler structure after the following variable substitution:

$$s = \frac{\sqrt{t-t'}}{\sqrt{t_1}} \qquad t - t' = t_1 s^2 \qquad \mathrm{d}t' = -2t_1 s\mathrm{d}s \tag{184}$$

Eqs.(181), (182), and (183) become after this substitution:

$$I_{ee}(r,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^{\sqrt{t/t_1}} \mathrm{e}^{-t/t_1+s^2} \cdot \mathrm{erf}\left(\frac{r}{sd_1}\right) 2s\mathrm{d}s \tag{185}$$

$$I_{eA}(p,r,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^{\sqrt{t/t_1}} \mathrm{e}^{-t/t_1+s^2} \cdot \left[\mathrm{erf}\left(\frac{r}{sd_1}\right) - \frac{r}{p} \cdot \exp\left(-\frac{r^2-p^2}{s^2 d_1^2}\right) \cdot \mathrm{erf}\left(\frac{p}{sd_1}\right)\right] 2s\mathrm{d}s$$

$$p = x,\ y \tag{186}$$

$$I_{eT}(x,y,z,t;1) = \frac{q_1 t_1}{e_0} \cdot \frac{u_0\sqrt{\pi}}{d_1} \int_0^{\sqrt{t/t_1}} \mathrm{e}^{-t/t_1+s^2} \cdot \mathrm{erf}\left(\frac{x}{sd_1}\right) \cdot \mathrm{erf}\left(\frac{y}{sd_1}\right) \cdot \exp\left(-\frac{z^2}{s^2 d_1^2}\right) 2\mathrm{d}s \tag{187}$$

Here, we have introduced the length:

$$d_1 = \sqrt{4at_1} \tag{188}$$

The first part of I_{eA} is the same integral as I_{ee}.

21.2 Displacement

The displacement components are given by Eqs. (171), (172), and (173):

$$u^{qe}(x,y,z,t;1) = I_e(t;1) \cdot \left[-\frac{u_0}{2} \ln\left(\frac{r+y}{r-y}\right) + u^{mb}(x,y,z)\right] +$$

$$+ u_0 \int_{1/\sqrt{4at}}^{\infty} I_e[t - 1/(4as^2); 1] \cdot \frac{\mathrm{erf}(ys)}{s} \cdot \mathrm{e}^{-(r^2-y^2)s^2} ds \tag{189}$$

$$v^{qe}(x,y,z,t;1) = I_e(t;1) \cdot \left[-\frac{u_0}{2} \ln\left(\frac{r+x}{r-x}\right) + v^{mb}(x,y,z)\right] +$$

$$+ u_0 \int_{1/\sqrt{4at}}^{\infty} I_e[t - 1/(4as^2); 1] \cdot \frac{\mathrm{erf}(xs)}{s} \cdot \mathrm{e}^{-(r^2-x^2)s^2} ds \tag{190}$$

$$w^{qe}(x,y,z,t;1) = I_e(t;1) \cdot \left[u_0 \arctan\left(\frac{xy}{zr}\right) + w^{mb}(x,y,z)\right] -$$

$$u_0 z\sqrt{\pi} \int_{1/\sqrt{4at}}^{\infty} I_e[t - 1/(4as^2); 1] \cdot \mathrm{erf}(xs) \cdot \mathrm{erf}(ys) \cdot \mathrm{e}^{-z^2 s^2} ds \tag{191}$$

The first factor I_e in the integrals becomes:

$$I_e[t - 1/(4as^2); 1] = \frac{q_1 t_1}{e_0} \cdot \left[1 - \exp\left(-\frac{t}{t_1} + \frac{1}{d_1^2 s^2}\right)\right] \tag{192}$$

21.3 Strain field

The components of the strain field caused by an exponentially decaying, quadrantal heat source become according to Eqs. (174):

$$\varepsilon_{xx}^{qe}(x,y,z,t;1) = \frac{u_0}{r} \cdot \frac{xy}{x^2+z^2} \cdot I_{eA}(y,r,t;1) + I_e(t;1) \cdot \varepsilon_{xx}^{mb}(x,y,z)$$

$$\varepsilon_{yy}^{qe}(x,y,z,t;1) = \frac{u_0}{r} \cdot \frac{xy}{y^2+z^2} \cdot I_{eA}(x,r,t;1) + I_e(t;1) \cdot \varepsilon_{yy}^{mb}(x,y,z)$$

$$\varepsilon_{zz}^{qe}(x,y,z,t;1) = -\frac{u_0}{r}\left[\frac{xy}{x^2+z^2}\cdot I_{eA}(y,r,t;1) + \frac{xy}{y^2+z^2}\cdot I_{eA}(x,r,t;1)\right] +$$

$$+ I_{eT}(x,y,z,t;1) + I_e(t;1)\cdot\varepsilon_{zz}^{mb}(x,y,z)$$

$$\varepsilon_{xz}^{qe}(x,y,z,t;1) = \frac{u_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot I_{eA}(y,r,t;1) + I_e(t;1)\cdot\varepsilon_{xz}^{mb}(x,y,z)$$

$$\varepsilon_{yz}^{qe}(x,y,z,t;1) = \frac{u_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot I_{eA}(x,r,t;1) + I_e(t;1)\cdot\varepsilon_{yz}^{mb}(x,y,z)$$

$$\varepsilon_{xy}^{qe}(x,y,z,t;1) = -\frac{u_0}{r}\cdot I_{ee}(r,t;1) + I_e(t;1)\cdot\varepsilon_{xy}^{mb}(x,y,z) \qquad (193)$$

21.4 Stress field

The components of the stress field caused by an exponentially decaying, quadrantal heat source become according to Eqs. (175):

$$\sigma_{xx}^{qe}(x,y,z,t;1) = \frac{p_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot I_{eA}(y,r,t;1) - \frac{E}{1+\nu}\cdot I_{eT}(x,y,z,t;1) + I_e(t;1)\cdot\sigma_{xx}^{mb}(x,y,z)$$

$$\sigma_{yy}^{qe}(x,y,z,t;1) = \frac{p_0}{r}\cdot\frac{xy}{y^2+z^2}\cdot I_{eA}(x,r,t;1) - \frac{E}{1+\nu}\cdot I_{eT}(x,y,z,t;1) + I_e(t;1)\cdot\sigma_{yy}^{mb}(x,y,z)$$

$$\sigma_{zz}^{qe}(x,y,z,t;1) = -\frac{p_0}{r}\left[\frac{xy}{x^2+z^2}\cdot I_{eA}(y,r,t;1) + \frac{xy}{y^2+z^2}\cdot I_{eA}(x,r,t;1)\right] + I_e(t;1)\cdot\sigma_{zz}^{mb}(x,y,z)$$

$$\sigma_{xz}^{qe}(x,y,z,t;1) = \frac{p_0}{r}\cdot\frac{yz}{x^2+z^2}\cdot I_{eA}(y,r,t;1) + I_e(t;1)\cdot\sigma_{xz}^{mb}(x,y,z)$$

$$\sigma_{yz}^{qe}(x,y,z,t;1) = \frac{p_0}{r}\cdot\frac{xz}{y^2+z^2}\cdot I_{eA}(x,r,t;1) + I_e(t;1)\cdot\sigma_{yz}^{mb}(x,y,z)$$

$$\sigma_{xy}^{qe}(x,y,z,t;1) = -\frac{p_0}{r}\cdot I_{ee}(r,t;1) + I_e(t;1)\cdot\sigma_{xy}^{mb}(x,y,z) \qquad (194)$$

21.5 Sum of exponentials

Let us now consider the case when $q(t)$ consists of a sum of exponentials:

$$q(t) = \sum_j q_j\cdot e^{-t/t_j} \qquad (195)$$

The solution for this case is readily obtained by superposition. We just add the solutions for q_1, t_1, for q_2, t_2, and so on. For example, the strain component ε_{xx} for the quadrantal heat source containing J exponentials is:

$$\varepsilon_{xx}(x,y,z,t) = \sum_{j=1}^{J}\left[\frac{u_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot I_{eA}(y,r,t;j) + I_e(t;j)\cdot\varepsilon_{xx}^{mb}(x,y,z)\right]$$

or, since the summation in j only involves the time-dependent integrals:

$$\varepsilon_{xx}(x,y,z,t) = \frac{u_0}{r}\cdot\frac{xy}{x^2+z^2}\cdot\sum_{j=1}^{J} I_{eA}(y,r,t;j) + \varepsilon_{xx}^{mb}(x,y,z)\cdot\sum_{j=1}^{J} I_e(t;j) \qquad (196)$$

The choice of the basic heat emission e_0 does not matter ($e_0 \neq 0$). The factor e_0 occurs in u_0 and in the integrals I_e, I_{eA}, I_{eT} and I_{ee}. It cancels in the formulas of Subsections 20.2-4.

21.6 General solution for rectangular, exponentially decaying heat source

The above solution is valid for a quadrantal, exponentially decaying heat source, $q(t) = q_1 \cdot e^{-t/t_1}$. The solution for a rectangular heat source is obtained by superposition of four quadrantal solutions with x replaced by $x \pm L$ and y by $y \pm B$. See Section 3 and in particular Eqs. (11) and (12). We have for any component f of the strain and stress (and displacement) fields:

$$f(x,y,z,t) = 0.25\,[f^{qe}(x+L,y+B,z,t;1) - f^{qe}(x+L,y-B,z,t;1)+$$

$$-f^{qe}(x-L,y+B,z,t;1) + f^{qe}(x-L,y-B,z,t;1)]$$

$$f = u,\ v,\ w,\ \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz},\ \varepsilon_{yz},\ \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz},\ \text{or}\ \sigma_{yz} \qquad (197)$$

Here, the displacement, strain and stress components (with upper index qe) are given by Eqs.(189-191), (193) and (194), respectively. The integrals I_{ee}, I_{eA} and I_{eT} in the formulas are given by Eqs. (185-187). The integral I_e is given by Eq. (180), and the constants u_0 and p_0 by Eq. (146). The time-independent displacement, strain and stress components, u^{mb} etc., $\varepsilon^{mb}_{..}$ and $\sigma^{mb}_{..}$, are defined by Eqs. (150), (152) and (154). The solution can be rewritten using the notation from Eq. (13):

$$f(x,y,z,t) = \sum_{n_x=\pm 1}\sum_{n_y=\pm 1} \frac{n_x n_y}{4} \cdot f^{qe}(x+n_x L, y+n_y B, z, t; 1)$$

$$f = u,\ v,\ w,\ \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz},\ \varepsilon_{yz},\ \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz},\ \text{or}\ \sigma_{yz} \qquad (198)$$

This solution concerns a single exponential heat source $q(t) = q_1 \cdot e^{-t/t_1}$, Eq. (177). The solution for a sum of exponential heat sources, Eq. (195), is obtained by adding the above type of solution for each component using the different constants q_j and t_j (and $d_j = \sqrt{4at_j}$). The solution for a sum of J exponentially decaying heat sources in a semi-infinite region is:

$$f(x,y,z,t) = \sum_{j=1}^{J}\sum_{n_x=\pm 1}\sum_{n_y=\pm 1} \frac{n_x n_y}{4} \cdot f^{qe}(x+n_x L, y+n_y B, z, t; j)$$

$$f = u,\ v,\ w,\ \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz},\ \varepsilon_{yz},\ \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz},\ \text{or}\ \sigma_{yz} \qquad (199)$$

22 The complete solution

We will in this section summarize the complete solution for a sum of exponential heat sources.

22.1 Problem and assumptions

The displacement $\mathbf{u} = (u, v, w)$ satisfies Navier's equation (2) in a semi-infinite, linearly elastic, isotropic, homogeneous medium:

$$\nabla^2(\mathbf{u}) + \frac{1}{1-2\nu}\nabla(e) = \frac{2\alpha(1+\nu)}{1-2\nu}\nabla T \qquad e = \nabla \cdot \mathbf{u}$$

$$-\infty < x < \infty \qquad -\infty < y < \infty \qquad -\infty < z < H \qquad (200)$$

The temperature field $T(x,y,z,t)$ is caused by a time-dependent heat source $q(t)$ (W/m^2) over a rectangular area: $-L < x < L$, $-B < y < B$, $z = 0$. The heat is emitted from the start $t = 0$. The heat source $q(t)$ consists of a sum of J exponentials:

$$q(t) = \sum_{j=1}^{J} q_j \cdot e^{-t/t_j} \tag{201}$$

The decay time constants t_j are any positive numbers.

The solution $(u, \varepsilon^{xx}, \sigma_{xx} \ldots)$ shall tend to zero at infinity ($\sqrt{x^2+y^2} \to \infty$ or $z \to -\infty$). The three stress components σ_{xz}, σ_{yz}, and σ_{zz} at the ground surface $z = H$ are zero. We have the boundary conditions, (5):

$$\sigma_{zz}(x,y,H,t) = 0 \quad \sigma_{zx}(x,y,H,t) = 0 \quad \sigma_{zy}(x,y,H,t) = 0 \tag{202}$$

The temperature T is zero at the ground surface, Eq. (61).

The above problem is solved in six steps described in Section 24. One main assumption is made in order to simplify the solving process. The assumption is that the far-field approximation of the quadrantal solution can be used at the ground surface. This is the case with good accuracy when t satisfies Eq. (63):

$$t < \frac{H^2}{16a} \tag{203}$$

In the KBS-3 example, t must be less than 300 years ($a = 1.62 \cdot 10^{-6}\text{m}^2/\text{s}$, $H = 500$ m), Eq. (68). After about 300 years the solution at the surface is disturbed. The disturbance will increase at the ground surface and will move deeper into the medium. The solution at the repository level is valid if:

$$t < \frac{H^2}{4a} \tag{204}$$

It will take four times longer for the disturbance to reach the repository level (roughly 1200 years) than the ground surface. The solution is totally incorrect for large times ($\simeq$10,000).

22.2 Parameters and auxiliary functions

The thermal and elastic properties of the rock are given by the volumetric heat capacity ρc ($\text{J/m}^3\text{K}$), the thermal diffusivity a (m/s), the coefficient of linear thermal expansion α (1/K), Poisson's ratio ν (-), and Young's modulus E (Pa). The length of the sides of the rectangular repository are given by $2L$ (m) and $2B$ (m), and the repository is situated at the distance H (m) under the ground surface in the plane $z = 0$. The heat emission of component j is described by the time constant t_j (s) and strength q_j (W/m^2). The value of e_0 may be chosen at will. Here, it is redundant.

The following parameters and auxiliary functions are used in the complete solution, Eqs. (146-148):

$$u_0 = \frac{1+\nu}{1-\nu} \cdot \frac{e_0\alpha}{\pi\rho c} \qquad p_0 = \frac{E}{1+\nu} \cdot u_0 = \frac{E}{1-\nu} \cdot \frac{e_0\alpha}{\pi\rho c}$$

$$r_m = \sqrt{x^2+y^2+(2H-z)^2} \qquad r = \sqrt{x^2+y^2+z^2}$$

$$D_x = x^2 + (2H-z)^2 \qquad D_y = y^2 + (2H-z)^2$$

$$B_x = 2(H-z)(2H-z)\left(\frac{1}{r_m^2} + \frac{2}{D_x}\right)$$

$$B_y = 2(H-z)(2H-z)\left(\frac{1}{r_m^2} + \frac{2}{D_y}\right)$$

$$d_j = \sqrt{4at_j} \tag{205}$$

22.3 Time integrals

The following dimensionless integrals are used in the final solution, Eqs. (180), (185), (186) and, (187):

$$I_e(t;j) = \int_0^t \tilde{q}(t')\mathrm{d}t' = \frac{q_j t_j}{e_0} \cdot (1 - \mathrm{e}^{-t/t_j}) \tag{206}$$

$$I_{ee}(r,t;j) = \frac{q_j t_j}{e_0} \cdot \int_0^{\sqrt{t/t_j}} \mathrm{e}^{-t/t_j+s^2} \cdot \mathrm{erf}\left(\frac{r}{sd_j}\right) 2s\mathrm{d}s \tag{207}$$

$$I_{eA}(p,r,t;j) = \frac{q_j t_j}{e_0} \cdot \int_0^{\sqrt{t/t_j}} \mathrm{e}^{-t/t_j+s^2} \cdot \left[\mathrm{erf}\left(\frac{r}{sd_j}\right) - \frac{r}{p} \cdot \exp\left(-\frac{r^2-p^2}{s^2 d_j^2}\right) \cdot \mathrm{erf}\left(\frac{p}{sd_j}\right)\right] 2s\mathrm{d}s$$

$$p = x,\ y \tag{208}$$

$$I_{eT}(x,y,z,t;j) = \frac{q_j t_j}{e_0} \cdot \frac{u_0\sqrt{\pi}}{d_j} \int_0^{\sqrt{t/t_j}} \mathrm{e}^{-t/t_j+s^2} \mathrm{erf}\left(\frac{x}{sd_j}\right) \mathrm{erf}\left(\frac{y}{sd_j}\right) \exp\left(-\frac{z^2}{s^2 d_j^2}\right) 2\mathrm{d}s \tag{209}$$

22.4 General formula for the solution

The solution is given by the general formula, (199):

$$f(x,y,z,t) = \sum_{j=1}^{J} \sum_{n_x=\pm 1} \sum_{n_y=\pm 1} \frac{n_x n_y}{4} \cdot f^{qe}(x + n_x L, y + n_y B, z, t; j) \tag{210}$$

The three displacement components are obtained for:

$$f = u,\ v \text{ or } w \tag{211}$$

The six components of the strain field are obtained for:

$$f = \varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz},\ \varepsilon_{xy},\ \varepsilon_{xz} \text{ or } \varepsilon_{yz} \tag{212}$$

The six components of the stress field are obtained for:

$$f = \sigma_{xx},\ \sigma_{yy},\ \sigma_{zz},\ \sigma_{xy},\ \sigma_{xz} \text{ or } \sigma_{yz} \tag{213}$$

22.5 Displacement

The three components of the displacement field for the quadrantal heat source $q_j \cdot \mathrm{e}^{-t/t_j}$ are given by Eqs. (189-191). Inserting the expressions (150) for the *mb* part, we get the following final explicit formulas:

$$u^{qe}(x,y,z,t;j) = I_e(t;j)\frac{u_0}{2}\left[-\ln\left(\frac{r+y}{r-y}\right) - (3-4\nu)\ln\left(\frac{r_m+y}{r_m-y}\right) + \frac{4y}{r_m} \cdot \frac{(H-z)(2H-z)}{x^2+(2H-z)^2}\right] +$$

$$+ u_0 \int_{1/\sqrt{4at}}^{\infty} I_e[t - 1/(4as^2); j] \cdot \frac{\mathrm{erf}(ys)}{s} \cdot \mathrm{e}^{-(r^2-y^2)s^2} ds \tag{214}$$

$$v^{qe}(x,y,z,t;j) = I_e(t;j)\frac{u_0}{2}\left[-\ln\left(\frac{r+x}{r-x}\right) - (3-4\nu)\ln\left(\frac{r_m+x}{r_m-x}\right) + \frac{4x}{r_m} \cdot \frac{(H-z)(2H-z)}{y^2+(2H-z)^2}\right] +$$

$$+ u_0 \int_{1/\sqrt{4at}}^{\infty} I_e[t - 1/(4as^2); j] \cdot \frac{\mathrm{erf}(xs)}{s} \cdot e^{-(r^2-x^2)s^2} ds \tag{215}$$

$$w^{qe}(x,y,z,t;j) = I_e(t;j)u_0 \left[\arctan\left(\frac{xy}{zr}\right) + (3-4\nu)\arctan\left(\frac{xy}{(2H-z)r_m}\right) + \right.$$

$$\left. + \frac{2xy}{r_m}\left(\frac{H-z}{x^2+(2H-z)^2} + \frac{H-z}{y^2+(2H-z)^2}\right)\right] -$$

$$u_0 z\sqrt{\pi} \int_{1/\sqrt{4at}}^{\infty} I_e[t - 1/(4as^2); j] \cdot \mathrm{erf}(xs) \cdot \mathrm{erf}(ys) \cdot e^{-z^2s^2} ds \tag{216}$$

22.6 Strains

The six components of the strain field for the quadrantal heat source $q_j \cdot e^{-t/t_j}$ are given by Eqs. (193):

$$\varepsilon_{xx}^{qe}(x,y,z,t;j) = \frac{u_0}{r} \cdot \frac{xy}{x^2+z^2} \cdot I_{eA}(y,r,t;j) + I_e(t;j) \cdot \varepsilon_{xx}^{mb}(x,y,z)$$

$$\varepsilon_{yy}^{qe}(x,y,z,t;j) = \frac{u_0}{r} \cdot \frac{xy}{y^2+z^2} \cdot I_{eA}(x,r,t;j) + I_e(t;j) \cdot \varepsilon_{yy}^{mb}(x,y,z)$$

$$\varepsilon_{zz}^{qe}(x,y,z,t;j) = -\frac{u_0}{r}\left[\frac{xy}{x^2+z^2} \cdot I_{eA}(y,r,t;j) + \frac{xy}{y^2+z^2} \cdot I_{eA}(x,r,t;j)\right] +$$

$$+ I_{eT}(x,y,z,t;j) + I_e(t;j) \cdot \varepsilon_{zz}^{mb}(x,y,z)$$

$$\varepsilon_{xz}^{qe}(x,y,z,t;j) = \frac{u_0}{r} \cdot \frac{yz}{x^2+z^2} \cdot I_{eA}(y,r,t;j) + I_e(t;j) \cdot \varepsilon_{xz}^{mb}(x,y,z)$$

$$\varepsilon_{yz}^{qe}(x,y,z,t;j) = \frac{u_0}{r} \cdot \frac{xz}{y^2+z^2} \cdot I_{eA}(x,r,t;j) + I_e(t;j) \cdot \varepsilon_{yz}^{mb}(x,y,z)$$

$$\varepsilon_{xy}^{qe}(x,y,z,t;j) = -\frac{u_0}{r} \cdot I_{ee}(r,t;j) + I_e(t;j) \cdot \varepsilon_{xy}^{mb}(x,y,z) \tag{217}$$

The time-independent parts $\varepsilon_{...}^{mb}$ are given by Eqs. (152):

$$\varepsilon_{xx}^{mb}(x,y,z) = \frac{u_0}{r_m} \cdot \frac{xy}{D_x}(3-4\nu-B_x)$$

$$\varepsilon_{yy}^{mb}(x,y,z) = \frac{u_0}{r_m} \cdot \frac{xy}{D_y}(3-4\nu-B_y)$$

$$\varepsilon_{zz}^{mb}(x,y,z) = \frac{u_0}{r_m}\left[\frac{xy}{D_x}(1-4\nu+B_x) + \frac{xy}{D_y}(1-4\nu+B_y)\right]$$

$$\varepsilon_{xy}^{mb}(x,y,z) = \frac{u_0}{r_m}\left[\frac{2(H-z)(2H-z)}{r_m^2} - 3 + 4\nu\right]$$

$$\varepsilon_{xz}^{mb}(x,y,z) = -\frac{u_0}{r_m}\left[\frac{y(4H-3z)}{D_x} - \frac{y(2H-z)}{D_x} \cdot B_x\right]$$

$$\varepsilon_{yz}^{mb}(x,y,z) = -\frac{u_0}{r_m}\left[\frac{x(4H-3z)}{D_y} - \frac{x(2H-z)}{D_y} \cdot B_y\right] \tag{218}$$

22.7 Stresses

The six components of the stress field for the quadrantal heat source $q_j \cdot e^{-t/t_j}$ are given by Eqs. (194)

$$\sigma_{xx}^{qe}(x,y,z,t;j) = \frac{p_0}{r} \cdot \frac{xy}{x^2+z^2} \cdot I_{eA}(y,r,t;j) - \frac{E}{1+\nu} \cdot I_{eT}(x,y,z,t;j) + I_e(t;j) \cdot \sigma_{xx}^{mb}(x,y,z)$$

$$\sigma_{yy}^{qe}(x,y,z,t;j) = \frac{p_0}{r} \cdot \frac{xy}{y^2+z^2} \cdot I_{eA}(x,r,t;j) - \frac{E}{1+\nu} \cdot I_{eT}(x,y,z,t;j) + I_e(t;j) \cdot \sigma_{yy}^{mb}(x,y,z)$$

$$\sigma_{zz}^{qe}(x,y,z,t;j) = -\frac{p_0}{r}\left[\frac{xy}{x^2+z^2} \cdot I_{eA}(y,r,t;j) + \frac{xy}{y^2+z^2} \cdot I_{eA}(x,r,t;j)\right] + I_e(t;j) \cdot \sigma_{zz}^{mb}(x,y,z)$$

$$\sigma_{xz}^{qe}(x,y,z,t;j) = \frac{p_0}{r} \cdot \frac{yz}{x^2+z^2} \cdot I_{eA}(y,r,t;j) + I_e(t;j) \cdot \sigma_{xz}^{mb}(x,y,z)$$

$$\sigma_{yz}^{qe}(x,y,z,t;j) = \frac{p_0}{r} \cdot \frac{xz}{y^2+z^2} \cdot I_{eA}(x,r,t;j) + I_e(t;j) \cdot \sigma_{yz}^{mb}(x,y,z)$$

$$\sigma_{xy}^{qe}(x,y,z,t;j) = -\frac{p_0}{r} \cdot I_{ee}(r,t;j) + I_e(t;j) \cdot \sigma_{xy}^{mb}(x,y,z) \tag{219}$$

The time-independent parts $\sigma_{...}^{mb}$ are given by Eqs. (154):

$$\sigma_{xx}^{mb}(x,y,z) = \frac{p_0}{r_m}\left[\frac{xy}{D_x}(3-B_x) + 4\nu\frac{xy}{D_y}\right]$$

$$\sigma_{yy}^{mb}(x,y,z) = \frac{p_0}{r_m}\left[\frac{xy}{D_y}(3-B_y) + 4\nu\frac{xy}{D_x}\right]$$

$$\sigma_{zz}^{mb}(x,y,z) = \frac{p_0}{r_m}\left[\frac{xy}{D_x}(1+B_x) + \frac{xy}{D_y}(1+B_y)\right]$$

$$\sigma_{xy}^{mb}(x,y,z) = \frac{p_0}{r_m}\left[\frac{2(H-z)(2H-z)}{r_m^2} - 3 + 4\nu\right]$$

$$\sigma_{xz}^{mb}(x,y,z) = -\frac{p_0}{r_m}\left[\frac{y(4H-3z)}{D_x} - \frac{y(2H-z)}{D_x} \cdot B_x\right]$$

$$\sigma_{yz}^{mb}(x,y,z) = -\frac{p_0}{r_m}\left[\frac{x(4H-3z)}{D_y} - \frac{x(2H-z)}{D_y} \cdot B_y\right] \tag{220}$$

23 Solution at the center

In this section formulas for the solution at the center ($x = 0$, $y = 0$ and $z = 0$) of the rectangular heat source are given. The general formula for one exponentially decaying component is, Eq. (210) with $J = 1$:

$$f(x,y,z,t) = \sum_{n_x=\pm1}\sum_{n_y=\pm1} \frac{n_x n_y}{4} \cdot f^{qe}(x+n_xL, y+n_yB, z, t; 1) \tag{221}$$

At the center $(x, y, z) = (0, 0, 0)$, the solution is given by:

$$f(0,0,0,t) = 0.25 \cdot [f^{qe}(L,B,0,t;1) - f^{qe}(-L,B,0,t;1)$$

$$-f^{qe}(L,-B,0,t;1) + f^{qe}(-L,-B,0,t;1)] \tag{222}$$

23.1 Symmetry relations

Now, if f^{qe} is an odd function with respect to x and y, then the above formula becomes:

$$f(0,0,0,t) = 0.25 \cdot [f^{qe}(L,B,0,t;1) + f^{qe}(L,B,0,t;1)+$$

$$+ f^{qe}(L,B,0,t;1) + f^{qe}(L,B,0,t;1)]$$

or

$$f(0,0,0,t) = f^{qe}(L,B,0,t;1) \qquad f^{qe} \text{ odd in } x \text{ and } y \tag{223}$$

For $f^{qe} = w^{qe}$, Eq. (216), the function is odd in both x and y, and we have in accordance with Eq. (223):

$$w(0,0,0,t) = w^{qe}(L,B,0,t;1) \tag{224}$$

The function $f(0,0,0,t)$ is zero as soon as f^{qe} is even with respect to x or y:

$$f(0,0,0,t) = 0 \qquad f^{qe} \text{ even in } x \text{ or } y \tag{225}$$

For example, for $f^{qe} = u^{qe}$, Eq. (214), which is even in x and odd in y, the solution at the center becomes:

$$u(0,0,0,t) = 0.25 \cdot [u^{qe}(L,B,0,t;1) - u^{qe}(L,B,0,t;1)+$$

$$+ u^{qe}(L,B,0,t;1) - u^{qe}(L,B,0,t;1)] = 0 \tag{226}$$

The only nonzero component of the displacement at the center is w.

From the expressions in Subsections 22.6 and 22.7 we see that all quadrantal shear strains and stresses are even in x or y, while the other three components are odd in x and y. The nonzero components of the strain and stress fields are ε_{xx}, ε_{yy}, ε_{zz}, σ_{xx}, σ_{yy}, and σ_{zz}.

23.2 Parameters at the center

The geometrical parameters of (205) are taken for $x = y = z = 0$. Using the index 0 to denote values at the center, we have:

$$r_{m0} = \sqrt{L^2 + B^2 + 4H^2} \qquad r_0 = \sqrt{L^2 + B^2}$$

$$D_{x0} = L^2 + 4H^2 \qquad D_{y0} = B^2 + 4H^2$$

$$B_{x0} = 4H^2 \left(\frac{1}{r_{m0}^2} + \frac{2}{D_{x0}} \right) \qquad B_{y0} = 4H^2 \left(\frac{1}{r_{m0}^2} + \frac{2}{D_{y0}} \right) \tag{227}$$

23.3 Time integrals at the center

The dimensionless time integrals of Subsection 22.3 become for the solution at the center:

$$I_e(t;1) = \frac{q_1 t_1}{e_0} \cdot (1 - e^{-t/t_1}) \tag{228}$$

$$I_{eA}(L,r_0,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^{\sqrt{t/t_1}} e^{-t/t_1+s^2} \cdot \left[\operatorname{erf}\left(\frac{r_0}{sd_1}\right) - \frac{r_0}{L} \cdot \exp\left(-\frac{B^2}{s^2 d_1^2}\right) \cdot \operatorname{erf}\left(\frac{L}{sd_1}\right)\right] 2s\mathrm{d}s$$

$$I_{eA}(B,r_0,t;1) = \frac{q_1 t_1}{e_0} \cdot \int_0^{\sqrt{t/t_1}} e^{-t/t_1+s^2} \cdot \left[\operatorname{erf}\left(\frac{r_0}{sd_1}\right) - \frac{r_0}{B} \cdot \exp\left(-\frac{L^2}{s^2 d_1^2}\right) \cdot \operatorname{erf}\left(\frac{B}{sd_1}\right)\right] 2s\mathrm{d}s$$

$$I_{eT}(L,B,0,t;1) = \frac{q_1 t_1}{e_0} \cdot \frac{u_0\sqrt{\pi}}{d_1} \int_0^{\sqrt{t/t_1}} e^{-t/t_1+s^2} \cdot \operatorname{erf}\left(\frac{L}{sd_1}\right) \cdot \operatorname{erf}\left(\frac{B}{sd_1}\right) 2\mathrm{d}s \qquad (229)$$

23.4 Displacement field at the center

The displacement field for a rectangular, exponentially decaying heat source has a comparatively simple form at the center $(x,y,z) = (0,0,0)$. Only the displacement component w is nonzero. The displacement components at the center become according to Eqs. (214)-(216), (223) and (225):

$$u(0,0,0,t;1) = 0 \qquad v(0,0,0,t;1) = 0$$

$$w(0,0,0,t;1) = u_0 \left[(3-4\nu)\arctan\left(\frac{LB}{2Hr_{m0}}\right) + \frac{2H}{r_{m0}}\left(\frac{LB}{D_{x0}} + \frac{LB}{D_{y0}}\right)\right] I_e(t;1) \qquad (230)$$

The component u^{qe} is even in x and the component v^{qe} is odd in y, and consequently these components give no contribution to the overall displacement at the center, Eqs. (225) and (226). The component w^{qe} is odd in both x and y and thus gives a contribution to $w(0,0,0,t;1)$ according to Eqs. (223) and (224).

23.5 Strain field at the center

The strain field for the rectangular, exponentially decaying heat source with a single component has a comparatively simple form at the center, $(x,y,z) = (0,0,0)$. We have from Eqs. (217) and (218):

$$\varepsilon_{xx}(0,0,0,t) = \frac{u_0}{r_0} \cdot \frac{B}{L} \cdot I_{eA}(B,r_0,t;1) + I_e(t;1) \cdot \frac{u_0}{r_{m0}} \cdot \frac{LB}{D_{x0}} [3-4\nu-B_{x0}]$$

$$\varepsilon_{yy}(0,0,0,t) = \frac{u_0}{r_0} \cdot \frac{L}{B} \cdot I_{eA}(L,r_0,t;1) + I_e(t;1) \cdot \frac{u_0}{r_{m0}} \cdot \frac{LB}{D_{y0}} [3-4\nu-B_{y0}]$$

$$\varepsilon_{zz}(0,0,0,t) = -\frac{u_0}{r_0}\left[\frac{B}{L} \cdot I_{eA}(B,r_0,t;1) + \frac{L}{B} \cdot I_{eA}(L,r_0,t;1)\right] + I_{eT}(L,B,0,t;1) +$$

$$+ I_e(t;1) \cdot \frac{u_0}{r_{m0}} \left[\frac{LB}{D_{x0}}(1-4\nu+B_{x0}) + \frac{LB}{D_{y0}}(1-4\nu+B_{y0})\right] +$$

$$\varepsilon_{xy}(0,0,0,t) = 0 \qquad \varepsilon_{xz}(0,0,0,t) = 0 \qquad \varepsilon_{yz}(0,0,0,t) = 0 \qquad (231)$$

23.6 Stress field at the center

The stress field for a rectangular, exponentially decaying heat source has, like the strain field, a simple form at the center $(x,y,z) = (0,0,0)$. The stress components at the center become from Eqs. (219) and (220):

$$\sigma_{xx}(0,0,0,t) = \frac{p_0}{r_0}\cdot\frac{B}{L}\cdot I_{eA}(B,r_0,t;1) - \frac{E}{1+\nu}\cdot I_{eT}(L,B,0,t;1)+$$

$$+ I_e(t;1)\cdot\frac{p_0}{r_{m0}}\left[\frac{LB}{D_{x0}}(3-B_{x0}) + 4\nu\frac{LB}{D_{y0}}\right]$$

$$\sigma_{yy}(0,0,0,t) = \frac{p_0}{r_0}\cdot\frac{L}{B}\cdot I_{eA}(L,r_0,t;1) - \frac{E}{1+\nu}\cdot I_{eT}(L,B,0,t;1)+$$

$$+ I_e(t;1)\cdot\frac{p_0}{r_{m0}}\left[\frac{LB}{D_{y0}}(3-B_{y0}) + 4\nu\frac{LB}{D_{x0}}\right]$$

$$\sigma_{zz}(0,0,0,t) = -\frac{p_0}{r_0}\left[\frac{B}{L}\cdot I_{eA}(B,r_0,t;1) + \frac{L}{B}\cdot I_{eA}(L,r_0,t;1)\right]+$$

$$+ I_e(t;1)\cdot\frac{p_0}{r_{m0}}\left[\frac{LB}{D_{x0}}(1+B_{x0}) + \frac{LB}{D_{y0}}(1+B_{y0})\right]$$

$$\sigma_{xy}(0,0,0,t) = 0 \qquad \sigma_{xz}(0,0,0,t) = 0 \qquad \sigma_{yz}(0,0,0,t) = 0 \tag{232}$$

23.7 Quadratic heat source area; $L = B$

The solution at the center $(0,0,0)$ is simplified somewhat for a quadratic heat source area; $L = B$. The integrals in Subsection 23.3 become with $r_0 = \sqrt{2}L$:

$$I_{eA}(L,\sqrt{2}L,t;1) = I_1(L/d_1,t/t_1) =$$

$$\frac{q_1 t_1}{e_0}\int_0^{\sqrt{t/t_1}} e^{-t/t_1+s^2}\cdot\left[\operatorname{erf}\left(\frac{\sqrt{2}L}{sd_1}\right) - \sqrt{2}\cdot\exp\left(-\frac{L^2}{d_1^2 s^2}\right)\cdot\operatorname{erf}\left(\frac{L}{sd_1}\right)\right]\cdot 2s\mathrm{d}s \tag{233}$$

$$I_{eT}(L,L,0,t;1) = I_2(L/d_1,t/t_1) = \frac{q_1 t_1}{e_0}\cdot\frac{u_0\sqrt{\pi}}{d_1}\int_0^{\sqrt{t/t_1}} e^{-t/t_1+s^2}\cdot\left[\operatorname{erf}\left(\frac{L}{sd_1}\right)\right]^2 2\mathrm{d}s \tag{234}$$

The integrals I_1 and I_2 depend of the dimensionless variables L/d_1 and t/t_1.
The displacements become according to Eqs. (230) and (227) with $L = B$:

$$u(0,0,0,t;1) = 0 \qquad v(0,0,0,t;1) = 0$$

$$w(0,0,0,t;1) = u_0\left[(3-4\nu)\arctan\left(\frac{L^2}{2H\sqrt{2L^2+4H^2}}\right)+\right.$$

$$\left.+\frac{2H}{\sqrt{2L^2+4H^2}}\cdot\frac{2L^2}{\sqrt{L^2+4H^2}}\right]\cdot I_e(t;1) \tag{235}$$

The strain components become according to Eqs. (231) and (227) with $L = B$:

$$\varepsilon_{xx}(0,0,0,t) = \frac{u_0}{L}\cdot\frac{1}{\sqrt{2}}\cdot I_1(L/d_1,t/t_1) + I_e(t;1)\cdot\frac{u_0}{\sqrt{2L^2+4H^2}}\cdot\frac{L^2}{L^2+4H^2}\cdot$$

$$\cdot\left[3-4\nu-4H^2\left(\frac{1}{2L^2+4H^2}+\frac{2}{L^2+4H^2}\right)\right] \tag{236}$$

$$\varepsilon_{yy}(0,0,0,t) = \varepsilon_{xx}(0,0,0,t) \tag{237}$$

$$\varepsilon_{zz}(0,0,0,t) = -\frac{u_0}{L}\cdot\sqrt{2}\cdot I_1(L/d_1,t/t_1) + I_2(L/d_1,t/t_1) + \frac{u_0}{\sqrt{2L^2+4H^2}}\cdot\frac{L^2}{L^2+4H^2}\cdot$$

$$\cdot 2\cdot\left[1-4\nu+4H^2\left(\frac{1}{2L^2+4H^2}+\frac{2}{L^2+4H^2}\right)\right]\cdot I_e(t;1) \tag{238}$$

$$\varepsilon_{xy}(0,0,0,t)=0 \qquad \varepsilon_{xz}(0,0,0,t)=0 \qquad \varepsilon_{yz}(0,0,0,t)=0 \tag{239}$$

The stress components become according to Eqs. (232) and (227) with $L = B$:

$$\sigma_{xx}(0,0,0,t) = \frac{p_0}{\sqrt{2}L}\cdot I_1(L/d_1,t/t_1) - \frac{E}{1+\nu}\cdot I_2(L/d_1,t/t_1)+$$

$$+\frac{p_0}{\sqrt{2L^2+4H^2}}\cdot\frac{L^2}{L^2+4H^2}\left[3-4H^2\left(\frac{1}{2L^2+4H^2}+\frac{2}{L^2+4H^2}\right)+4\nu\right]\cdot I_e(t;1) \tag{240}$$

$$\sigma_{yy}(0,0,0,t) = \sigma_{xx}(0,0,0,t) \tag{241}$$

$$\sigma_{zz}(0,0,0,t) = -\frac{p_0}{L}\cdot\sqrt{2}\cdot I_1(L/d_1,t/t_1) + \frac{p_0}{\sqrt{2L^2+4H^2}}\cdot\frac{2L^2}{L^2+4H^2}\cdot$$

$$\cdot\left[1+4H^2\left(\frac{1}{2L^2+4H^2}+\frac{2}{L^2+4H^2}\right)\right]\cdot I_e(t;1) \tag{242}$$

$$\sigma_{xy}(0,0,0,t)=0 \qquad \sigma_{xz}(0,0,0,t)=0 \qquad \sigma_{yz}(0,0,0,t)=0 \tag{243}$$

24 Survey of the solution procedure

The solution presented in this report contains many parts, and it is quite intricate. A survey of the different steps in the analysis may be appropriate. The final solution is obtained by superposition involving six steps:

1. Solution for an instantaneous, quadrantal heat source in an infinite region (Infinite solution). See Sections 3-11.

2. Solution for an instantaneous, quadrantal mirror heat source at $z = 2H$ (Mirror solution). See Section 13.

3. Solution of the remaining boundary-value problem in the semi-infinite region $-\infty < z < H$ (Boundary solution). See Sections 14-18.

4. Solution for a single, exponentially decaying heat source via a Duhamel superposition of the instantaneous solution. See Sections 20-21.

5. Solutions for a sum of exponentially decaying heat sources by superposition. See Subsection 21.5.

6. Solution for the rectangular heat source by superposition of four quadrantal solutions. See Subsection 21.6.

The thermoelastic process is induced by a rectangular heat source. The problem involves the length and width of the rectangle. A crucial step to facilitate the analysis is the introduction of the quadrantal heat source, Eq. (6). The solution for the rectangular source is obtained by superposition of four quadrantal solutions, Eqs. (13) and (176). The length and width do not occur in the quadrantal problem.

The solution for the instantaneous, quadrantal heat source at $z = 0$, Eq. (6), in infinite space $-\infty < x, y, z < \infty$ is presented in Sections 3-11. The temperature field T_{qi} from the quadrantal heat source, Eq. (8), is calculated in Section 3. The displacements induced by the instantaneous, quadrantal heat source are solutions to Navier's Eq. (2) with $T = T_{qi}$. A displacement potential Φ is used in the solution. The problem is reduced to the Poisson equation (16), where the Laplacian of Φ is equal to the temperature (times a constant). In Section 5, the problem is formulated in dimensionless form. The dimensionless problem for the instantaneous quadrantal heat source in infinite space does not involve *any* free dimensionless parameters. The solution Φ_q depends only on the dimensionless spacial coordinates, Eq. (22). The time t only occurs in the form $\sqrt{4at}$ as a scale factor. The displacement potential Φ_q is calculated in Section 6. It is given by a double integral of $\text{erf}(r)/r$, Eq. (41). The solution may be expressed as a single integral with a somewhat more complicated integrand involving powers, error functions and an exponential, Eq. (42).

The three displacement components u, v and w are obtained by the first derivatives of the displacement potential. The final expressions for the displacement due to the quadrantal heat source in infinite space are Eqs. (46). The three single integrals must be evaluated numerically. The strains and stresses are obtained from second-order derivatives of the displacement potential. A gratifying fact is that the integrals may be evaluated by partial integrations. The final explicit expressions for strain and stress fields are Eqs. (50) and (52), respectively. The far-field approximation $z' = z/\sqrt{4at} \gg 1$ is studied in Section 11. The solution is simplified considerably. The displacement, strain, and stress fields become time-independent, Eqs. (56), (58) and (59). The far-field approximation is valid with good accuracy for $z' = z/\sqrt{4at} > 2$.

The corrections to the solution in infinite space in order to satisfy the boundary conditions (5) at the ground surface are discussed in Section 12. The corrections are derived under the simplifying assumption that the far-field solution may be used at the ground surface, Eq. (63). (The conditions on the exact solution are discussed in Appendix 1.) The solution is valid everywhere for $t < H^2/(16a)$. With typical KBS-3 data (H = 500 m), the time limit becomes around 300 years. The error develops slowly from the ground surface. The solution is valid at the repository level $z = 0$ (and downwards) during a four times longer period ($t <$ 1200 years).

In the second step, a mirror instantaneous, quadrantal heat source at $z = 2H$ is introduced, Eq. (69). The far-field approximation may be used. The mirror solution is given by Eqs. (71), (73) and (74) in Section 13.

The quadrantal and mirror solutions are added. The boundary conditions are considered in Section 14. The three components of the stress field σ_{xz}, σ_{yz}, and σ_{zz}, should all be zero at the ground surface $z = H$, Eqs. (5). The two shear stresses are zero because of symmetric heat sources at $z = 0$ and $z = 2H$. The normal stress σ_{zz} is given by Eq. (76). The remaining problem for the instantaneous quadrantal heat source is to solve a boundary-value problem in the semi-infinite space $-\infty < z < H$. There is a time-independent normal stress over the boundary, while the shear stresses are zero.

This boundary value problem is studied in Sections 14-18. General formulas for the solution due to Hertz are given in Section 15. The solution is obtained by derivatives up to the third order of a certain potential χ. This Hertzian potential is given by a double integral involving the prescribed normal stress over the boundary surface $z = H$, Eq. (90). We have succeeded in solving the quite intricate integral in the way described in Section 16. The beautiful formula (112) for χ is obtained. The boundary problem is formulated and solved in the region $0 < z < \infty$. The resulting boundary solution (with index $b0$) is given in Appendix 2. The solution is then transformed, Eqs. (87-89), to the actual region $-\infty < z < H$ in Section 18. The total solution for the instantaneous quadrantal heat source is given by the sum the above three solutions. The complete result is given in Section 19.

The solution for any time-dependent quadrantal heat source $q(t)$ (W/m^2) in the semi-infinite space $-\infty < z < H$ is discussed in Section 20. The solution is obtained by a Duhamel integral involving q and the quadrantal solution, Eq. (157). The displacement is given by Eqs. (171-173). It is noteworthy that only single integrals occur. The strain and stress fields are given by Eqs. (174) and (175). Finally, the solution for the rectangular heat source is obtained by superposition of four quadrantal solutions, Eqs. (176).

The particular case of a single, exponentially decaying heat source, Eq. (177), is dealt with in Section 21. The solution is given in Subsections 21.1-4. The solution for a sum of exponential sources, Eq. (195), over the rectangular area is obtained by superposition of four quadrantal solutions, Eqs. (199).

The complete solution for a sum of exponential heat sources is summarized in Section 22.

The solution at the center (0,0,0) is of special interest. It is presented in detail in Section 23.

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Appendix 1. Conditions for solution for large times

The solution presented above is derived under the assumption (63). This means that the far-field approximation of the quadrantal solution may be used at the ground surface. The temperature field caused by the heat source at $z = 0$ has not reached the ground surface $z = H$. We will here present the conditions on the solution without this restriction.

We start as before with the basic quadrantal solution for an infinite medium. The total temperature at the boundary is zero. This means that we need a mirror heat source with the *opposite* sign at $z = 2H$. The source at $z = 0$, Eq. (6), and mirror source at $z = 2H$ with opposite sign are given by our quadrantal solution minus the same solution with z replaced by $z - 2H$.

The stress components from these two parts (with upper index $q - m$) become at the ground surface:

$$\sigma_{zz}^{q-m}(x, y, H, t) = 0 \tag{244}$$

$$\sigma_{xz}^{q-m}(x, y, H, t) = \frac{2p_0}{\sqrt{x^2 + y^2 + H^2}} \cdot \frac{yH}{x^2 + H^2} \cdot A'\left(\frac{y}{\sqrt{4at}}, \frac{\sqrt{x^2 + y^2 + H^2}}{\sqrt{4at}}\right) \tag{245}$$

$$\sigma_{yz}^{q-m}(x, y, H, t) = \frac{2p_0}{\sqrt{x^2 + y^2 + H^2}} \cdot \frac{xH}{y^2 + H^2} \cdot A'\left(\frac{x}{\sqrt{4at}}, \frac{\sqrt{x^2 + y^2 + H^2}}{\sqrt{4at}}\right) \tag{246}$$

Now , the normal stress vanishes, while the shear stresses from source and mirror source are equal. Here, A is defined by Eq. (48).

The boundary solution shall satisfy the three above time-dependent boundary conditions. The total quadrantal solution consists of the basic quadrantal solution for infinite medium minus the same solution for the quadrantal heat source at $z = 2H$. The third part is the boundary solution for the above boundary conditions with nonzero shear stresses.

Appendix 2. Boundary solution in $z > 0$

The boundary solution for the region $z > 0$ is obtained by insertion of the derivatives of χ from Section 17 in the formulas (91), (92) and (95). This solution is denoted by index $b0$.

The three displacement components are from Eqs. (91), (114-115), (117), and (119-120):

$$u^{b0} = 2u_0 \left[\frac{z}{r_H} \cdot \frac{y(z + H)}{x^2 + (z + H)^2} - \frac{1 - 2\nu}{2} \ln\left(\frac{r_H + y}{r_H - y}\right)\right] \tag{247}$$

$$v^{b0} = 2u_0 \left[\frac{z}{r_H} \cdot \frac{x(z + H)}{y^2 + (z + H)^2} - \frac{1 - 2\nu}{2} \ln\left(\frac{r_H + x}{r_H - x}\right)\right] \tag{248}$$

$$w^{b0} = -2u_0 \left[\frac{z}{r_H} \left(\frac{xy}{x^2 + (z + H)^2} + \frac{xy}{y^2 + (z + H)^2}\right) + \right.$$

$$\left. +2(1 - \nu) \arctan\left(\frac{xy}{(z + H) r_H}\right)\right] \tag{249}$$

Here, we have also used that $p_0 = 2\mu u_0$, Eq. (35). The strain field is given by Eqs. (92) and the derivatives (116-126):

$$\varepsilon_{xx}^{b0} = 2u_0 \cdot \frac{1}{r_H} \cdot \frac{xy}{x^2 + (z + H)^2} \left(1 - 2\nu - z(z + H)\left(\frac{1}{r_H^2} + \frac{2}{x^2 + (z + H)^2}\right)\right) \tag{250}$$

$$\varepsilon_{yy}^{b0} = 2u_0 \cdot \frac{1}{r_H} \cdot \frac{xy}{y^2 + (z+H)^2} \left(1 - 2\nu - z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{y^2 + (z+H)^2}\right)\right) \quad (251)$$

$$\varepsilon_{zz}^{b0} = 2u_0 \cdot \frac{1}{r_H} \left[\frac{xy}{x^2 + (z+H)^2} \left(1 - 2\nu + z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{x^2 + (z+H)^2}\right)\right) + \frac{xy}{y^2 + (z+H)^2} \left(1 - 2\nu + z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{y^2 + (z+H)^2}\right)\right)\right] \quad (252)$$

$$\varepsilon_{xy}^{b0} = 2u_0 \cdot \frac{1}{r_H} \left[\frac{z(z+H)}{r_H^2} - 1 + 2\nu\right] \quad (253)$$

$$\varepsilon_{xz}^{b0} = 2u_0 \cdot \frac{1}{r_H} \cdot \frac{yz}{x^2 + (z+H)^2} \left(1 - (z+H)^2\left(\frac{1}{r_H^2} + \frac{2}{x^2 + (z+H)^2}\right)\right) \quad (254)$$

$$\varepsilon_{yz}^{b0} = 2u_0 \cdot \frac{1}{r_H} \cdot \frac{xz}{y^2 + (z+H)^2} \left(1 - (z+H)^2\left(\frac{1}{r_H^2} + \frac{2}{y^2 + (z+H)^2}\right)\right) \quad (255)$$

The stress field is given by Eqs. (95) and the derivatives (116-126):

$$\sigma_{xx}^{b0} = 2p_0 \cdot \frac{1}{r_H} \left[\frac{xy}{x^2 + (z+H)^2} \left(1 - z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{x^2 + (z+H)^2}\right)\right) + +2\nu \frac{xy}{y^2 + (z+H)^2}\right] \quad (256)$$

$$\sigma_{yy}^{b0} = 2p_0 \cdot \frac{1}{r_H} \left[\frac{xy}{y^2 + (z+H)^2} \left(1 - z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{y^2 + (z+H)^2}\right)\right) + +2\nu \frac{xy}{x^2 + (z+H)^2}\right] \quad (257)$$

$$\sigma_{zz}^{b0} = 2p_0 \cdot \frac{1}{r_H} \left[\frac{xy}{x^2 + (z+H)^2} \left(1 + z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{x^2 + (z+H)^2}\right)\right) + + \frac{xy}{y^2 + (z+H)^2} \left(1 + z(z+H)\left(\frac{1}{r_H^2} + \frac{2}{y^2 + (z+H)^2}\right)\right)\right] \quad (258)$$

$$\sigma_{xy}^{b0} = \frac{p_0}{u_0} \cdot \varepsilon_{xy}^{b0} \qquad \sigma_{xz}^{b0} = \frac{p_0}{u_0} \cdot \varepsilon_{xz}^{b0} \qquad \sigma_{yz}^{b0} = \frac{p_0}{u_0} \cdot \varepsilon_{yz}^{b0} \quad (259)$$

Nomenclature

The following abbreviations are used in the nomenclature list:

b.s.	boundary solution
comp.	component(s)
dim.	dimensionless
displ.	displacement
dist.	distance
exp.dec.	exponentially decaying
h.s.	heat source
inst.	instantaneous
quad.	quadrantal
rect.	rectangular
s-i.reg.	semi-infinite region
t-d.	time-dependent
t-i.	time-independent

a	$= \lambda/(\rho c)$, thermal diffusivity of the rock	(m^2/s)
$A'(x', r')$	auxiliary function, Eq. (48)	(-)
$A(p, r, t)$	auxiliary function, Eq. (51), ($p = x$, y)	(-)
B	length of the rect. h.s. in the y-direction	(m)
B_x	auxiliary function, Eq. (147)	(-)
B_{x0}	auxiliary function, Eq. (227)	(-)
B_y	auxiliary function, Eq. (148)	(-)
B_{y0}	auxiliary function, Eq. (227)	(-)
c	specific heat capacity of the rock	(J/kgK)
$c\rho$	volumetric heat capacity of the rock	(J/m^3K)
d_j	$= \sqrt{4at_j}$, Eq. (188)	(m)
D	canister spacing in repository tunnels	(m)
D'	dist. between repository tunnels	(m)
D_x	$= x^2 + (2H - z)^2$, auxiliary function, Eq. (129)	(m^2)
D_{x0}	$= L^2 + 4H^2$, auxiliary function, Eq. (227)	(m^2)
D_y	$= y^2 + (2H - z)^2$, auxiliary function, Eq. (129)	(m^2)
D_{y0}	$= B^2 + 4H^2$, auxiliary function, Eq. (227)	(m^2)
e	exponential function	(-)
e	volume expansion, Eq. (2)	(-)
e_0	inst. heat emission per unit area at $t = 0$, Eq. (6)	(J/m^2)
e_b	volume expansion for the boundary solution, Eq. (83)	(-)
e_{b0}	volume expansion for the boundary solution, Eq. (85)	(-)
erf	error function, Eq. (9)	(-)
exp	exponential function	(-)
E	Young's modulus, Eq. (4)	(Pa)
H	dist. from the h.s. to the ground surface	(m)
I_e	time integral, Eq. (180)	(-)
I_{ee}	time integral, Eq. (185)	(-)

I_{eA}	time integral, Eq. (186)	(-)
I_{eT}	time integral, Eq. (187)	(-)
I_q	time integral, Eq. (158)	(-)
I_{qe}	time integral, Eq. (159)	(-)
I_{qA}	time integral, Eq. (160)	(-)
I_{qT}	time integral, Eq. (162)	(-)
I_1	time integral, Eq. (233)	(-)
I_2	time integral, Eq. (234)	(-)
L	length of the rect. h.s. in the x-direction	(m)
p_0	scale factor for stress comp., Eq. (35)	(Pa·m)
$q(t)$	time-dependent h.s., Eq. (156)	(W/m^2)
$q(t)$	time-dependent heat emission per canister	(W/m^2)
q_1	heat emission per unit area at $t = 0$ for comp. 1, Eq. (177)	(W/m^2)
q_j	inst. heat emission per unit area at $t = 0$ for comp. j, Eq. (195)	(W/m^2)
$\tilde{q}(t)$	time-dependence of heat emission, Eq. (156)	(s^{-1})
r	$= \sqrt{x^2 + y^2 + z^2}$	(m)
r_0	$= \sqrt{L^2 + B^2}$, Eq. (227)	(m)
r_H	$= \sqrt{x^2 + y^2 + (z + H)^2}$, Eq. (113)	(m)
r_{0H}	$= \sqrt{x^2 + y^2 + H^2}$, Eq. (77)	(m)
r_m	$= \sqrt{x^2 + y^2 + (z - 2H)^2}$, Eq. (72)	(m)
r_{m0}	$= \sqrt{L^2 + B^2 + 4H^2}$, Eq. (227)	(m)
r'	$= r/\sqrt{4at}$ dim. r-coordinate	(-)
t	time	(s)
t_1	time-constant for component 1, Eq. (177)	(s)
t_j	time-constant for exponential component j, Eq. (195)	(s)
$T_{rec,i}$	temperature field for the rect. h.s., Eq. (12)	(K)
T_q	dim. quad. inst. temp. field, Eq. (25)	(-)
T_{qi}	quad. inst. temp. field, Eq. (8)	(K)
u, v, w	displ. comp.	(m)
u^q, v^q, w^q	dim. displ. comp. (quad. inst. h.s.), Eq. (31)	(-)
u^{qe}, v^{qe}, w^{qe}	displ. comp. (exp.dec. quad. inst. h.s. s-i.reg.), Eqs. (189-191)	(m)
u^{qi}, v^{qi}, w^{qi}	t-d. displ. comp. (quad. inst. h.s.), Eq. (46)	(m)
u^{qq}, v^{qq}, w^{qq}	displ. comp. (t-d. quad. inst. h.s. s-i.reg.), Eq. (157)	(m)
u^{qs}, v^{qs}, w^{qs}	displ. comp. (quad. inst. h.s. s-i.reg.), Eq. (145)	(m)
u^{mb}, v^{mb}, w^{mb}	t-i. displ. comp. (quad. inst. h.s.), Eq. (150)	(m)
u^m, v^m, w^m	displ. comp. (mirror h.s.), Eq. (71)	(m)
u^b, v^b, w^b	displ. comp. (boundary solution), Eq. (130)	(m)
$\mathbf{u}$	displ., Eq. (15)	(m)
$\mathbf{u}_q$	dim. displ., Eq. (30)	(-)
$\mathbf{u}_b$	displ. for the boundary solution, Eq. (83)	(m)
$\mathbf{u}_{b0}$	displ. for the boundary solution, Eq. (85)	(m)
u_0	scale factor for displ., Eq. (28)	(m)
x, y, z	Cartesian coordinates	(m)
x'	$= x/\sqrt{4at}$, dim. x-coordinate, Eq. (22)	(-)
y'	$= y/\sqrt{4at}$, dim. y-coordinate, Eq. (22)	(-)
z'	$= z/\sqrt{4at}$ dim. z-coordinate, Eq. (22)	(-)

α	coefficient of linear thermal expansion	(1/K)
$\varepsilon_{xx} \cdots$	strain comp.	(-)

$\varepsilon^{q}_{xx}\cdots$	dim. strain comp. (quad. inst. h.s.), Eq. (33)	(-)
$\varepsilon^{qe}_{xx}\cdots$	strain comp. (exp.dec. quad. inst. h.s.), Eq. (193)	(-)
$\varepsilon^{qi}_{xx}\cdots$	t-d. strain comp. (quad. inst. h.s.), Eq. (50)	(-)
$\varepsilon^{qq}_{xx}\cdots$	strain comp. (t-d. quad. inst. h.s.), Eq. (157)	(-)
$\varepsilon^{qs}_{xx}\cdots$	strain comp. (quad. inst. h.s. s-i.reg), Eq. (145)	(-)
$\varepsilon^{mb}_{xx}\cdots$	t-i. strain comp. (quad. inst. h.s.), Eq. (152)	(-)
$\varepsilon^{m}_{xx}\cdots$	strain comp. (mirror h.s.), Eq. (73)	(-)
$\varepsilon^{b}_{xx}\cdots$	strain comp. (b.s.), Eq. (133)	(-)
χ	Hertz' potential, Eq. (90)	(-)
μ	Shear modulus, Eq. (4)	(-)
ν	Poisson's ratio, Eq. (4)	(-)
ρ	density of the rock	(kg/m^3)
$\sigma_{xx}\cdots$	stress comp.	(Pa)
$\sigma^{q}_{xx}\cdots$	dim. stress comp. (quad. inst. h.s.), Eq. (36)	(-)
$\sigma^{qe}_{xx}\cdots$	stress comp. (exp.dec. quad. inst. h.s.), Eq. (194)	(Pa)
$\sigma^{qi}_{xx}\cdots$	t-d. stress comp. (quad. inst. h.s.), Eq. (52)	(Pa)
$\sigma^{qq}_{xx}\cdots$	stress comp. (t-d. quad. inst. h.s.), Eq. (157)	(Pa)
$\sigma^{qs}_{xx}\cdots$	stress comp. (quad. inst. h.s. s-i.reg.), Eq. (145)	(Pa)
$\sigma^{mb}_{xx}\cdots$	t-i. stress comp. (quad. inst. h.s.), Eq. (154)	(Pa)
$\sigma^{m}_{xx}\cdots$	stress comp. (mirror h.s.), Eq. (74)	(Pa)
$\sigma^{b}_{xx}\cdots$	stress comp. (b.s.), Eq. (139)	(Pa)
σ_{b0}	scale factor for stress comp. at ground surface, Eq. (79)	(Pa)
σ'_{b}	dim. stress comp. at ground surface, Eq. (80)	(-)
$\sigma'_{b,max}$	maximum dim. stress comp. at ground surface, Eq. (81)	(-)
Φ	displ. potential, Eq. (15)	(m^2)
Φ_q	dim. displ. potential, Eq. (27)	(-)

A prime indicates that the quantity is dimensionless.

List of indices

q Dimensionless instantaneous quadrantal solution

qi Infinite instantaneous quadrantal solution

m Mirror solution

b Boundary solution

mb Sum of mirror and boundary solution

qe Semi-infinite exponentially decaying, quadrantal solution

qs Semi-infinite instantaneous quadrantal solution

qq Semi-infinite time-dependent, quadrantal solution

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[2] University of Göteborg, Department of General and Marine Microbiology, Göteborg, Sweden
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